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Soliton solutions for ABS lattice equations: I. Cauchy matrix approach

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Abstract

In recent years there have been new insights into the integrability of quadrilateral lattice equations, i.e. partial difference equations which are the natural discrete analogues of integrable partial differential equations in 1+1 dimensions. In the scalar (i.e. single-field) case, there now exist classification results by Adler, Bobenko and Suris (ABS) leading to some new examples in addition to the lattice equations 'of KdV type' that were known since the late 1970s and early 1980s. In this paper, we review the construction of soliton solutions for the KdV-type lattice equations and use those results to construct *N*-soliton solutions for all lattice equations in the ABS list except for the elliptic case of Q4, which is left to a separate treatment.

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1. Introduction

The study of integrable partial difference equations (PΔEs) dates back to the pioneering work of Ablowitz and Ladik, [1], and of Hirota, [17], motivated partly by the search for numerical finite-difference schemes that are discrete in time as well as in space. Such systems constitute discrete analogues of soliton-type partial differential equations (PDEs). In subsequent work the Lie-algebraic approach of the Kyoto school, [13], on the one hand, and the approach using a so-called *direct linearization* method, cf [23, 28], on the other hand led to new systematic constructions for such systems. In recent years, they have been reinvestigated from various points of view, including reductions to integrable dynamical mappings [26, 27], and associated finite-gap solutions [10, 14, 21], reductions to discrete Painlevé equations [22, 24] and similarity reductions, whilst soliton solutions arose as a direct corollary from the original constructions mentioned above.

The property of multidimensional consistency, first set out explicitly in [25], cf also [11], lies implicitly within lattice equations which have the interpretation of a superposition principle for Bäcklund transformations (BT). This was identified in [24] as the property constituting the precise discrete analogue of the existence of hierarchies of nonlinear evolution equations, and hence of integrability. The property has subsequently been used by Adler, Bobenko and Suris (ABS) [3, 4] as a classifying property. Within certain additional conditions they produced a full list of scalar quadrilateral lattice equations which are multidimensionally consistent. This list, which surprisingly is quite short, is reminiscent of Painlevé's list of transcendental equations in the case of second-order ordinary differential equations (ODEs) possessing the property of non-movable singularities of the general solution. We reproduce this list from [3] below[4] containing three groups of equations: the Q-list, the H-list and the A-list:

Q-list:

$$\text{Q1}: \quad \mathring{p}(u-\widehat{u})(\widetilde{u}-\widehat{\widetilde{u}}) - \mathring{q}(u-\widetilde{u})(\widehat{u}-\widehat{\widetilde{u}}) = \delta^2 \mathring{p}\mathring{q}(\mathring{q}-\mathring{p}) \tag{1.1a}$$

$$\begin{aligned}\text{Q2}: \quad & \mathring{p}(u-\widehat{u})(\widetilde{u}-\widehat{\widetilde{u}}) - \mathring{q}(u-\widetilde{u})(\widehat{u}-\widehat{\widetilde{u}}) + \mathring{p}\mathring{q}(\mathring{p}-\mathring{q})(u+\widetilde{u}+\widehat{u}+\widehat{\widetilde{u}}) \\ & = \mathring{p}\mathring{q}(\mathring{p}-\mathring{q})(\mathring{p}^2-\mathring{p}\mathring{q}+\mathring{q}^2)\end{aligned} \tag{1.1b}$$

$$\begin{aligned}\text{Q3}: \quad & \mathring{p}(1-\mathring{q}^2)(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}}) - \mathring{q}(1-\mathring{p}^2)(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}}) \\ & = (\mathring{p}^2-\mathring{q}^2)\left((\widehat{u}\widetilde{u}+u\widehat{\widetilde{u}}) + \delta^2\frac{(1-\mathring{p}^2)(1-\mathring{q}^2)}{4\mathring{p}\mathring{q}}\right)\end{aligned} \tag{1.1c}$$

$$\begin{aligned}\text{Q4}: \quad & \mathring{p}(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}}) - \mathring{q}(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}}) \\ & = \frac{\mathring{p}\mathring{Q}-\mathring{q}\mathring{P}}{1-\mathring{p}^2\mathring{q}^2}\left((\widehat{u}\widetilde{u}+u\widehat{\widetilde{u}}) - \mathring{p}\mathring{q}(1+u\widetilde{u}\widehat{u}\widehat{\widetilde{u}})\right)\end{aligned} \tag{1.1d}$$

where $\mathring{P}^2 = \mathring{p}^4 - \gamma\mathring{p}^2 + 1$, $\mathring{Q}^2 = \mathring{q}^4 - \gamma\mathring{q}^2 + 1$.

H-list:

$$\text{H1}: \quad (u-\widehat{\widetilde{u}})(\widetilde{u}-\widehat{u}) = \mathring{p}-\mathring{q} \tag{1.2a}$$

$$\text{H2}: \quad (u-\widehat{\widetilde{u}})(\widetilde{u}-\widehat{u}) = (\mathring{p}-\mathring{q})(u+\widetilde{u}+\widehat{u}+\widehat{\widetilde{u}}) + \mathring{p}^2-\mathring{q}^2 \tag{1.2b}$$

$$\text{H3}: \quad \mathring{p}(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}}) - \mathring{q}(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}}) = \delta(\mathring{q}^2-\mathring{p}^2). \tag{1.2c}$$

A-list:

$$\text{A1}: \quad \mathring{p}(u+\widehat{u})(\widetilde{u}+\widehat{\widetilde{u}}) - \mathring{q}(u+\widetilde{u})(\widehat{u}+\widehat{\widetilde{u}}) = \delta^2\mathring{p}\mathring{q}(\mathring{p}-\mathring{q}) \tag{1.3a}$$

$$\text{A2}: \quad \mathring{p}(1-\mathring{q}^2)(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}}) - \mathring{q}(1-\mathring{p}^2)(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}}) + (\mathring{p}^2-\mathring{q}^2)(1+u\widetilde{u}\widehat{u}\widehat{\widetilde{u}}) = 0. \tag{1.3b}$$

The notation we have adopted here and in earlier papers is the following: the vertices along an elementary plaquette on a rectangular lattice contain the dependent variables:

$$\begin{aligned} u &:= u_{n,m}, & \widetilde{u} &:= u_{n+1,m}, \\ \widehat{u} &:= u_{n,m+1}, & \widehat{\widetilde{u}} &:= u_{n+1,m+1}, \end{aligned}$$

which schematically are indicated in figure 1, in which $\mathring{p}$ and $\mathring{q}$ denote *lattice parameters* associated with the directions in the lattice (measuring the grid size in these directions):

[4] Note that the Q4 equation as given by [3], which was first found by Adler in [2], is different from that in (1.1*d*), which in this form was first presented in [15].

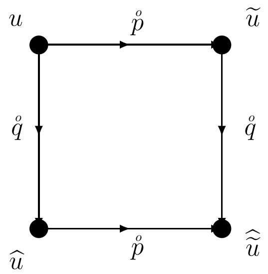

Figure 1. Arrangement of the shifted dependent variable on the vertices of a quadrilateral and association of the lattice parameters with the edges.

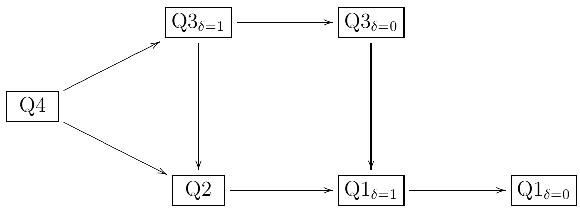

Figure 2. Coalescence diagram for equations in the ABS Q-list (this was first given by Adler and Suris in [5]).

$u \overset{\overset{o}{p}}{\to} \widetilde{u}$, $u \overset{\overset{o}{q}}{\to} \widehat{u}$. The lattice parameters associated with the two directions on the lattice play a central role in the notion of multidimensional consistency. Specifically, they parametrize the family of equations which are compatible on the multidimensional lattice. The equations in the Q-list are related by degeneration (of the elliptic curve associated with Q4) through the coalescence scheme illustrated in figure 2. The H-list and the A-list emerge also from degeneration from the Q-list, leading, in principle, to an extension of figure 2, and we will make some of these connections explicitly in the later part of the present paper. Connections between the ABS equations are not limited to coalescence by degeneration; there also exist Miura- and Bäcklund-type transformations connecting distinct equations in the list; some transformations of Bäcklund type were discussed recently in [9].

The present paper is part I of a sequence of papers dedicated to closed-form *N*-soliton solutions of the lattice equations mentioned above. Part II, by Hietarinta and Zhang [16], will be dedicated to the Casorati form of the same solutions, establishing a different methodology. The results in the present paper are based on a Cauchy matrix structure which goes back to earlier work [23, 28], on the so-called *direct linearization approach*. In section 2, we review the application of that approach to soliton solutions of lattice equations 'of Korteweg–de Vries (KdV) type', which were cases already known in the earlier papers mentioned. Specifically, these include the lattice potential KdV, potential modified KdV, Schwarzian KdV and an interpolating equation between them which has been referred to as the NQC equation (cf [29]). (These equations are equivalent to the equations H1, $H3_{\delta=0}$, $Q1_{\delta=0}$ and $Q3_{\delta=0,}$ respectively, up to gauge transformations.) The machinery introduced in section 2 will comprise several objects, and relations between them, which we will need throughout the remainder of the paper, because the *N*-soliton solution for the full Q3 equation can be expressed in terms of those quantities. A constructive proof of this *N*-soliton solution is presented in section 3, whilst in section 4 we show that this N-soliton solution is consistent with the BT (i.e., defining the latter by a copy of the lattice equation itself we establish the precise relation between the N- and the $N+1$-soliton solution in terms of this BT). As an upshot of the present treatment, it becomes apparent that the natural parametrization of Q3 turns out to involve already an elliptic curve, whose branch points can be viewed as lattice parameters associated with additional lattice directions. Degeneration from Q3 in this parametrization by bringing together one or more branch points of the curve yields one or other of the equations on the ABS list. In section 5, the degenerations from this parametrization of Q3 are given in detail and we construct the N-soliton solutions for all the equations in the ABS list, except for the elliptic case of Q4. This reveals, on the level of the N-soliton solutions, a deep connection between the equations expressed in terms of the basic quantities introduced in section 2. Section 6 contains a discussion of the results and concluding remarks.

2. *N*-soliton solutions of KdV-type lattice equations

Among the equations in the ABS list, we distinguish a subclass which we call of KdV type, and they comprise the lattice equations which already appeared many years ago in the context of discretizations of the KdV equations and its counterparts, cf, e.g., [17, 23, 28]. In this section, we introduce objects from which we construct solutions of the KdV-type lattice equations.

2.1. Cauchy matrix and recurrence structure

We will start by introducing the following Cauchy-type matrix:

$$\boldsymbol{M} = (M_{i,j})_{i,j=1,\dots,N}, \qquad M_{i,j} \equiv \frac{\rho_i c_j}{k_i + k_j}, \tag{2.1}$$

which will form the core of the structure which we will develop. In (2.1), the c_i, k_i, $(i = 1, \dots, N)$ denote two sets of N nonvanishing parameters, which we may choose freely (apart from that we have to assume that $k_i + k_j \neq 0, \forall\ i, j = 1, \dots, N$, in order to avoid difficulties with the numerators in the matrix $\boldsymbol{M}$). These parameters are assumed not to depend on the lattice variables, i.e. on the discrete independent variables n and m. The dependence on the latter is incorporated wholly in the functions ρ_i, the so-called *plane-wave factors*, which have the form

$$\rho_i = \left(\frac{p+k_i}{p-k_i}\right)^n \left(\frac{q+k_i}{q-k_i}\right)^m \rho_i^0, \tag{2.2}$$

where the ρ_i^0 are initial values, i.e. constant with regard to the variables n, m.[5]

[5] Importantly, the ρ_i^0 can still contain an, in principle, arbitrary number of additional discrete exponential factors of the form given in (2.2), i.e. depending on additional lattice variables, each of which would be associated with its own lattice parameter. Thus, a more general form for the ρ_i would be

$$\rho_i = \prod_{\substack{\nu \\ p_\nu \neq \pm k_i}} \left(\frac{p_\nu + k_i}{p_\nu - k_i}\right)^{n_\nu},$$

containing an arbitrary number of lattice variables n_ν each with the lattice parameter p_ν labelled by some index ν. Every statement derived below can be extended without restriction to involve any choice of these variables, and in particular this implies the multidimensional consistency of the equations derived from the scheme. Thus, we will say that these results can be *covariantly extended* to the multidimensional case.

Let us now introduce, for convenience, the following notation. Let $\boldsymbol{K}$ denote the diagonal $N \times N$ matrix containing the parameters k_i on the diagonal and introduce a column vector $\boldsymbol{r}$, containing the entries ρ_i, and a row vector ${}^t\boldsymbol{c}$, containing the entries c_i, i.e.

$$\boldsymbol{K} = \begin{pmatrix} k_1 & & & \\ & k_2 & & \\ & & \ddots & \\ & & & k_N \end{pmatrix}, \qquad \boldsymbol{r} = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_N \end{pmatrix}, \qquad {}^t\boldsymbol{c} = (c_1, c_2, \ldots, c_N)\,. \tag{2.3}$$

It is easily checked that we have from definition (2.1) immediately the relation

$$\boldsymbol{M}\boldsymbol{K} + \boldsymbol{K}\boldsymbol{M} = \boldsymbol{r}\,{}^t\boldsymbol{c}, \tag{2.4}$$

where significantly the *dyadic* on the right-hand side is a matrix of rank 1.

Next we establish the dynamics in terms of the matrix $\boldsymbol{M}$, which follows from definition (2.1) together with the dynamical equations for ρ_i (2.2) which are simply

$$\widetilde{\rho}_i = T_p\rho_i = \rho_i(n+1,m) = \frac{p+k_i}{p-k_i}\rho_i, \qquad \widehat{\rho}_i = T_q\rho_i = \rho_i(n,m+1) = \frac{q+k_i}{q-k_i}\rho_i, \tag{2.5}$$

where T_p, T_q denote the elementary shift operators in the lattice in the directions associated with lattice parameters p and q, respectively, $T_{-p} = T_p^{-1}$ and $T_{-q} = T_q^{-1}$ denoting their inverses. A straightforward calculation then yields the following relations:

$$\widetilde{\boldsymbol{M}}(p\boldsymbol{1}+\boldsymbol{K}) - (p\boldsymbol{1}+\boldsymbol{K})\boldsymbol{M} = \widetilde{\boldsymbol{r}}\,{}^t\boldsymbol{c}, \tag{2.6a}$$

$$\widehat{\boldsymbol{M}}(q\boldsymbol{1}+\boldsymbol{K}) - (q\boldsymbol{1}+\boldsymbol{K})\boldsymbol{M} = \widehat{\boldsymbol{r}}\,{}^t\boldsymbol{c}, \tag{2.6b}$$

in which $\boldsymbol{1}$ is the $N \times N$ unit matrix, and where we have used the obvious notation that the shifts $\widetilde{\ }$, $\widehat{\ }$ act on all the relevant objects depending on n, m by the respective shifts by one unit in these independent variables. In addition, we have the adjoint relations

$$(p\boldsymbol{1}-\boldsymbol{K})\widetilde{\boldsymbol{M}} - \boldsymbol{M}(p\boldsymbol{1}-\boldsymbol{K}) = \boldsymbol{r}\,{}^t\boldsymbol{c}, \tag{2.6c}$$

$$(q\boldsymbol{1}-\boldsymbol{K})\widehat{\boldsymbol{M}} - \boldsymbol{M}(q\boldsymbol{1}-\boldsymbol{K}) = \boldsymbol{r}\,{}^t\boldsymbol{c}. \tag{2.6d}$$

Equations (2.6) encode all the information on the dynamics of the matrix $\boldsymbol{M}$, w.r.t. the discrete variables n, m, in addition to (2.4) which can be thought of as the defining property of $\boldsymbol{M}$.

Now we introduce several objects involving the matrix $\boldsymbol{M}$, in terms of which we can define the basic variables which will solve the relevant lattice equations. Thus, we introduce the determinant

$$f = f_{n,m} = \det(\boldsymbol{1}+\boldsymbol{M}), \tag{2.7}$$

which we will identify later as the relevant τ-function. (We remark that the determinant (2.7) is similar to that considered in the work by Hirota, cf, e.g., [18], where it is referred to as 'Grammian solution' because of the method employed for the construction. We, however, prefer to call such solutions 'of Cauchy type', because the construction here is different.) Furthermore, we introduce the following quantities:

$$\boldsymbol{u}^{(i)} = (\boldsymbol{1}+\boldsymbol{M})^{-1}\boldsymbol{K}^i\boldsymbol{r}, \tag{2.8a}$$

$${}^t\boldsymbol{u}^{(j)} = {}^t\boldsymbol{c}\boldsymbol{K}^j(\boldsymbol{1}+\boldsymbol{M})^{-1}, \tag{2.8b}$$

$$S^{(i,j)} = {}^t\boldsymbol{c}\boldsymbol{K}^j(\boldsymbol{1}+\boldsymbol{M})^{-1}\boldsymbol{K}^i\boldsymbol{r}, \tag{2.8c}$$

for $i, j \in \mathbb{Z}$, assuming that none of the parameters k_i is zero[6]. Thus, we obtain an infinite sequence of column vectors $\boldsymbol{u}^{(i)}$, of row vectors ${}^t\boldsymbol{u}^{(j)}$ and an infinite by infinite array of scalar quantities $S^{(i,j)}$. An important property of the latter objects, which can also be written as

$$S^{(i,j)} = {}^t\boldsymbol{c}\boldsymbol{K}^j\boldsymbol{u}^{(i)} = {}^t\boldsymbol{u}^{(j)}\boldsymbol{K}^i\boldsymbol{r}, \tag{2.9}$$

is that they are symmetric w.r.t. the interchange of the indices, i.e.

$$S^{(i,j)} = S^{(j,i)}, \tag{2.10}$$

provided that the constants c_i, ρ_i^0, $k = 1, \ldots, N$, are all nonzero.

We shall now derive, starting from (2.6), a system of recurrence relations which describe the dynamics for the quantities defined in (2.8). Once this recursive structure is established, we will single out specific combinations of the $S^{(i,j)}$ in terms of which we can derive closed-form discrete equations. In fact, from definition (2.8*a*), using the fact that $\boldsymbol{K}$ is a diagonal matrix, we have

$$\begin{aligned}
\boldsymbol{K}^i\boldsymbol{r} = (\boldsymbol{1}+\boldsymbol{M})\boldsymbol{u}^{(i)} &\Rightarrow \boldsymbol{K}^i\widetilde{\boldsymbol{r}} = (\boldsymbol{1}+\widetilde{\boldsymbol{M}})\widetilde{\boldsymbol{u}}^{(i)} \\
&\Rightarrow \boldsymbol{K}^i\frac{p\boldsymbol{1}+\boldsymbol{K}}{p\boldsymbol{1}-\boldsymbol{K}}\boldsymbol{r} = (\boldsymbol{1}+\widetilde{\boldsymbol{M}})\widetilde{\boldsymbol{u}}^{(i)} \\
&\Rightarrow \boldsymbol{K}^i(p\boldsymbol{1}+\boldsymbol{K})\boldsymbol{r} = (p\boldsymbol{1}-\boldsymbol{K})(\boldsymbol{1}+\widetilde{\boldsymbol{M}})\widetilde{\boldsymbol{u}}^{(i)} \\
&= (p\boldsymbol{1}-\boldsymbol{K})\widetilde{\boldsymbol{u}}^{(i)} + (p\boldsymbol{1}-\boldsymbol{K})\widetilde{\boldsymbol{M}}\widetilde{\boldsymbol{u}}^{(i)} \\
&= (p\boldsymbol{1}-\boldsymbol{K})\widetilde{\boldsymbol{u}}^{(i)} + \boldsymbol{M}(p\boldsymbol{1}-\boldsymbol{K})\widetilde{\boldsymbol{u}}^{(i)} + \boldsymbol{r}\,{}^t\boldsymbol{c}\widetilde{\boldsymbol{u}}^{(i)},
\end{aligned}$$

where in the last step use has been made of (2.6*c*). Using now (2.8*c*) we conclude that

$$p\boldsymbol{K}^j\boldsymbol{r} + \boldsymbol{K}^{j+1}\boldsymbol{r} = (\boldsymbol{1}+\boldsymbol{M})(p\boldsymbol{1}-\boldsymbol{K})\widetilde{\boldsymbol{u}}^{(i)} + \widetilde{S}^{(i,0)}\boldsymbol{r}.$$

Multiplying both sides by the inverse matrix $(\boldsymbol{1}+\boldsymbol{M})^{-1}$ and identifying the terms on the left-hand side using (2.8*a*), we thus obtain

$$\begin{aligned}
(p\boldsymbol{1}-\boldsymbol{K})\widetilde{\boldsymbol{u}}^{(i)} &= (\boldsymbol{1}+\boldsymbol{M})^{-1}[p\boldsymbol{K}^i\boldsymbol{r} + \boldsymbol{K}^{i+1}\boldsymbol{r} - \widetilde{S}^{(i,0)}\boldsymbol{r}] \\
&= p\boldsymbol{u}^{(i)} + \boldsymbol{u}^{(i+1)} - \widetilde{S}^{(i,0)}\boldsymbol{u}^{(0)}.
\end{aligned} \tag{2.11}$$

Thus, we have obtained a linear recursion relation between the objects $\boldsymbol{u}^{(i)}$ with the objects $S^{(i,j)}$ acting as coefficients. In quite a similar fashion we can derive the relation

$$(p\boldsymbol{1}+\boldsymbol{K})\boldsymbol{u}^{(i)} = p\widetilde{\boldsymbol{u}}^{(i)} - \widetilde{\boldsymbol{u}}^{(i+1)} + S^{(i,0)}\widetilde{\boldsymbol{u}}^{(0)}, \tag{2.12}$$

in fact by making use of (2.6*c*) in this case. Equation (2.12) can be thought of as an inverse relation to (2.11), noting that the $\widetilde{\ }$-shifted objects are now at the right-hand side of the equation. Multiplying both sides of either (2.11) or (2.12) from the left by the row vector ${}^t\boldsymbol{c}\boldsymbol{K}^j$ and identifying the resulting terms through (2.8*c*), we obtain now a relation purely in terms of the objects $S^{(i,j)}$, namely

$$\begin{aligned}
{}^t\boldsymbol{c}\boldsymbol{K}^j(p\boldsymbol{1}-\boldsymbol{K})\widetilde{\boldsymbol{u}}^{(i)} &= {}^t\boldsymbol{c}\boldsymbol{K}^j[p\boldsymbol{u}^{(i)} + \boldsymbol{u}^{(i+1)} - \widetilde{S}^{(i,0)}\boldsymbol{u}^{(0)}] \\
\Rightarrow \quad p\widetilde{S}^{(i,j)} - \widetilde{S}^{(i,j+1)} &= pS^{(i,j)} + S^{(i+1,j)} - \widetilde{S}^{(i,0)}S^{(0,j)},
\end{aligned} \tag{2.13}$$

using the fact that $\widetilde{S}^{(i,0)}$ is just a scalar factor which can be moved to the left of the matrix multiplication. Thus, we have now obtained a *nonlinear* recursion relation between the $S^{(i,j)}$ and its $\widetilde{\ }$-shifted counterparts.

[6] Here and earlier we use the symbols ${}^t\boldsymbol{c}$ and ${}^t\boldsymbol{u}$ to denote *adjoint* row vectors, which in the case of the first is just the transpose of the column vector $\boldsymbol{c}$, i.e. ${}^t\boldsymbol{c} = \boldsymbol{c}^T$, but in the case of ${}^t\boldsymbol{u}$ is not simply the transpose of the column vector $\boldsymbol{u}$, but rather a quantity that is defined in (2.8*b*) in its own right. Thus, the left super-index 't' should not be confused with the operation of transposition, but rather indicate a new object obeying some linear equations associated with the vector $\boldsymbol{u}$.

In a similar fashion, multiplying equation (2.12) by the row vector ${}^t\boldsymbol{c}\boldsymbol{K}^j$ we obtain the complementary relation:

$$pS^{(i,j)} + S^{(i,j+1)} = p\widetilde{S}^{(i,j)} - \widetilde{S}^{(i+1,j)} + S^{(i,0)}\widetilde{S}^{(0,j)}. \tag{2.14}$$

However, this relation can be obtained from the previous relation (2.13) by interchanging the indices, using the symmetry (2.10).

Remark. Combining both equations (2.11) and (2.12), using also (2.12), the following *algebraic* (i.e. not involving lattice shifts) recurrence relation can be derived:

$$\boldsymbol{K}^2\boldsymbol{u}^{(i)} = \boldsymbol{u}^{(i+2)} + S^{(i,1)}\boldsymbol{u}^{(0)} - S^{(i,0)}\boldsymbol{u}^{(1)}, \tag{2.15}$$

which, in turn, gives rise to the following algebraic recurrence for the objects $S^{(i,j)}$, namely

$$S^{(i,j+2)} = S^{(i+2,j)} + S^{(i,1)}S^{(0,j)} - S^{(i,0)}S^{(1,j)}. \tag{2.16}$$

To summarize the structure obtained, we note that starting from the Cauchy matrix $\boldsymbol{M}$ defined in (2.1), depending dynamically on the lattice variables through the plane-wave factors ρ_i given in (2.2), we have defined an infinite set of objects, namely column and row vectors $\boldsymbol{u}^{(i)}$ and ${}^t\boldsymbol{u}^{(i)}$, and a doubly infinite sequence of scalar functions $S^{(i,j)}$, all related through a system of dynamical (since it involves lattice shift) recurrence relations. Obviously, all relations that we have derived for the $\widetilde{\ }$-shifts (involving the lattice parameter p and the lattice variable n) hold also for the $\widehat{\ }$-shifts, simply by replacing p by q and interchanging the roles of n and m. Thus, for the scalar objects $S^{(i,j)}$ we have the following set of coupled recurrence relations:

$$p\widetilde{S}^{(i,j)} - \widetilde{S}^{(i,j+1)} = pS^{(i,j)} + S^{(i+1,j)} - \widetilde{S}^{(i,0)}S^{(0,j)}, \tag{2.17a}$$

$$q\widehat{S}^{(i,j)} - \widehat{S}^{(i,j+1)} = qS^{(i,j)} + S^{(i+1,j)} - \widehat{S}^{(i,0)}S^{(0,j)}, \tag{2.17b}$$

We proceed now by deriving closed-form lattice equations for individual elements chosen from the $S^{(i,j)}$ as functions of the variables n, m.

2.2. *Closed-form lattice equations*

We start with the variable $S^{(0,0)}$, for which we can derive a partial difference equation as follows. In fact, subtracting (2.17*b*) from (2.17*a*) we obtain

$$p\widetilde{S}^{(i,j)} - q\widehat{S}^{(i,j)} - \widetilde{S}^{(i,j+1)} + \widehat{S}^{(i,j+1)} = (p-q)S^{(i,j)} - (\widetilde{S}^{(i,0)} - \widehat{S}^{(i,0)})S^{(0,j)}. \tag{2.18}$$

On the other hand, taking the $\widehat{\ }$-shift of (2.17*a*), let us refer to it as $\widehat{2.17a}$, and subtracting from it the $\widetilde{\ }$-shift of (2.17*b*), i.e. $\widetilde{2.17b}$, we obtain

$$p\widehat{S}^{(i,j)} - q\widetilde{S}^{(i,j)} + \widehat{S}^{(i,j+1)} - \widetilde{S}^{(i,j+1)} = (p-q)\widehat{\widetilde{S}}_{(i,j)} + (\widehat{S}^{(i,0)} - \widetilde{S}^{(i,0)})\widehat{\widetilde{S}}^{(0,j)}. \tag{2.19}$$

Combining both equations, the terms which have a shift in their second index drop out and we obtain the equation

$$(p+q)(\widetilde{S}^{(i,j)} - \widehat{S}^{(i,j)}) = (p-q)(S^{(i,j)} - \widehat{\widetilde{S}}^{(i,j)}) + (\widehat{S}^{(i,0)} - \widetilde{S}^{(i,0)})(S^{(0,j)} - \widehat{\widetilde{S}}^{(0,j)}). \tag{2.20}$$

Setting now $i = j = 0$ in the last formula, we see that we get a closed-form equation in terms of $w \equiv S^{(0,0)}$. This reads, after some trivial algebra, the equation

$$(p + q + w - \widehat{\widetilde{w}})(p - q + \widehat{w} - \widetilde{w}) = p^2 - q^2, \tag{2.21}$$

which is the lattice potential KdV equation, which has appeared in the literature in various guises, cf [17, 23, 28], notably as the permutability condition of the BTs for the potential KdV equation, cf [30]. Curiously, this integrable partial difference equation can also be traced back to numerical analysis, where it has appeared in the form of the ϵ-algorithm of Wynn, [32], as an efficient convergence accelerator algorithm. In the present context of the structures arising from the Cauchy matrix, we have established here an infinite family of solutions of the form

$$w \equiv S^{(0,0)} = {}^t\boldsymbol{c}(\boldsymbol{1}+\boldsymbol{M})^{-1}\boldsymbol{r}, \tag{2.22}$$

which constitute the N-soliton solutions for equation (2.21).

Equation (2.21) is by no means the only equation that emerges from the set of relations (2.17). In fact, instead of singling out $w = S^{(0,0)}$, we can choose other elements among the $S^{(i,j)}$, or (linear) combinations of them, and then systematically investigate what equations these choices satisfy by exploring the system of recurrence relations (2.17). For example, from (2.18) taking $i = 0$, $j = -1$ and introducing the variable $v \equiv 1 - S^{(0,-1)}$ it is a simple exercise to obtain the following relation:

$$p - q + \widehat{w} - \widetilde{w} = \frac{p\widetilde{v} - q\widehat{v}}{v}. \tag{2.23}$$

Alternatively, adding the $\widehat{\ }$-shift of (2.17*a*) to (2.17*b*), we get

$$p\widehat{\widetilde{S}}^{(i,j)} + qS^{(i,j)} - \widehat{\widetilde{S}}^{(i,j+1)} + S^{(i,j+1)} = (p+q)\widehat{S}^{(i,j)} + (S^{(i,0)} - \widehat{\widetilde{S}}^{(i,0)})\widehat{S}^{(0,j)}, \tag{2.24}$$

and when taking in (2.24) $i = 0$, $j = -1$ an easy calculation yields:

$$p + q + w - \widehat{\widetilde{w}} = \frac{p\widehat{\widetilde{v}} + qv}{\widehat{v}}. \tag{2.25}$$

Clearly, in (2.25), interchanging p and q, and the $\widetilde{\ }$-shift and the $\widehat{\ }$-shift should not make a difference, since the left-hand side is invariant under this change. Thus, the right-hand side must be invariant as well, leading to the relation

$$p(v\widehat{v} - \widetilde{v}\widehat{\widetilde{v}}) = q(v\widetilde{v} - \widehat{v}\widehat{\widetilde{v}}). \tag{2.26}$$

Equation (2.26) is an integrable PΔE in its own right for the quantity v, for which, by construction, we have an infinite family of solutions, namely given by

$$v = 1 - S^{(0,-1)} = 1 - {}^t\boldsymbol{c}\boldsymbol{K}^{-1}(\boldsymbol{1}+\boldsymbol{M})^{-1}\boldsymbol{r}. \tag{2.27}$$

The PΔE (2.26) for the variable v is identified as the lattice potential modified KdV (MKdV) equation, which also occurred in [23, 28], and is actually closely related to the lattice sine-Gordon equation of [17]. Relations (2.23) and (2.25) constitute a Miura transform between the lattice potential MKdV (2.26) and the lattice potential KdV equation (2.21).

As another choice of dependent variables we can consider the variable $S^{(-1,-1)}$, i.e. we can consider (2.17) for $i = j = -1$, leading to

$$p(\widetilde{S}^{(-1,-1)} + S^{(-1,-1)}) = 1 - (1 - \widetilde{S}^{(-1,0)})(1 - S^{(0,-1)}),$$

and a similar relation for p replaced by q and the $\widetilde{\ }$-shift replaced by the $\widehat{\ }$-shift. Using the fact that $S^{(-1,0)} = S^{(0,-1)} = 1 - v$ and introducing the abbreviation $z = S^{(-1,-1)} - \frac{n}{p} - \frac{m}{q}$, the latter relations reduce to

$$p(z - \widetilde{z}) = \widetilde{v}v, \qquad q(z - \widehat{z}) = \widehat{v}v. \tag{2.28}$$

On the one hand, these two equations lead back to equation (2.26) by eliminating the variable z (considering, in addition, the $\widetilde{\ }$- and $\widehat{\ }$-shifts of the two relations). On the other hand, by eliminating the variable v we obtain yet again a PΔE, but now for z which reads

$$\frac{(z - \widetilde{z})(\widehat{z} - \widehat{\widetilde{z}})}{(z - \widehat{z})(\widetilde{z} - \widehat{\widetilde{z}})} = \frac{q^2}{p^2}, \tag{2.29}$$

which we identify with the Schwarzian lattice KdV equation, and is also referred to as *cross-ratio equation*[7]. Here we have constructed N-soliton solutions of equation (2.29) given explicitly by the formula

$$z = {}^t\boldsymbol{c}\boldsymbol{K}^{-1}(\mathbf{1}+\boldsymbol{M})^{-1}\boldsymbol{K}^{-1}\boldsymbol{r} - z_0 - \frac{n}{p} - \frac{m}{q}, \tag{2.30}$$

in which z_0 is an arbitrary constant.

To summarize, we conclude that through the recurrence structure encoded in equations (2.17) we obtain solutions of various different PΔEs in one stroke. We will now proceed further by establishing a large parameter class of additional lattice equations, which also provide us information on the bilinear structure of the lattice systems.

2.3. Bilinear equations and the NQC equation

We will start by considering the τ-function for the soliton solutions, given by (2.7), using relation (2.6*a*), we can perform the following straightforward calculation:

$$\begin{aligned}\widetilde{f} &= \det(\mathbf{1}+\widetilde{\boldsymbol{M}}) = \det\{\mathbf{1}+[(p\mathbf{1}+\boldsymbol{K})\boldsymbol{M}+\widetilde{\boldsymbol{r}}\,{}^t\boldsymbol{c}](p\mathbf{1}+\boldsymbol{K})^{-1}\}\\ &= \det\{(p\mathbf{1}+\boldsymbol{K})[\mathbf{1}+\boldsymbol{M}+(p\mathbf{1}+\boldsymbol{K})^{-1}\widetilde{\boldsymbol{r}}\,{}^t\boldsymbol{c}](p\mathbf{1}+\boldsymbol{K})^{-1}\}\\ &= \det\{(\mathbf{1}+\boldsymbol{M})[\mathbf{1}+(\mathbf{1}+\boldsymbol{M})^{-1}(p\mathbf{1}+\boldsymbol{K})^{-1}\widetilde{\boldsymbol{r}}\,{}^t\boldsymbol{c}]\}\\ &= f\det\{\mathbf{1}+(\mathbf{1}+\boldsymbol{M})^{-1}(p\mathbf{1}+\boldsymbol{K})^{-1}\widetilde{\boldsymbol{r}}\,{}^t\boldsymbol{c}\},\end{aligned}$$

from which, using also

$$\widetilde{\boldsymbol{r}} = \frac{p\mathbf{1}+\boldsymbol{K}}{p\mathbf{1}-\boldsymbol{K}}\boldsymbol{r},$$

we have

$$\frac{\widetilde{f}}{f} = 1 + {}^t\boldsymbol{c}(\mathbf{1}+\boldsymbol{M})^{-1}(p\mathbf{1}-\boldsymbol{K})^{-1}\boldsymbol{r} = 1 + {}^t\boldsymbol{u}^{(0)}(p\mathbf{1}-\boldsymbol{K})^{-1}\boldsymbol{r}, \tag{2.31}$$

where in the last step we have made use of the determinant relation (a special case of the famous Weinstein–Aronszajn formula),

$$\det(\mathbf{1}+\boldsymbol{x}\boldsymbol{y}^T) = 1 + \boldsymbol{y}^T\cdot\boldsymbol{x},$$

for arbitrary N-component vectors $\boldsymbol{x}$, $\boldsymbol{y}$ (the superscript T denoting transposition).

The combination emerging on the right-hand side of equation (2.31) is a new object which we need in the scheme, and it is natural to try and derive some equations for it along the lines of the derivations in subsection 2.1. Thus, more generally, let us introduce the function

$$V(a) \equiv 1 - {}^t\boldsymbol{c}(a+K)^{-1}\boldsymbol{u}^{(0)} = 1 - {}^t\boldsymbol{u}^{(0)}(a\mathbf{1}+\boldsymbol{K})^{-1}\boldsymbol{r}, \tag{2.32}$$

for any value of a parameter $a \in \mathbb{C}$. From (2.31) we immediately have

$$\frac{\widetilde{f}}{f} = \frac{T_p f}{f} = V(-p), \tag{2.33}$$

but we need further relations involving $V(a)$ to derive closed-form equations for the τ-function f. To do that we introduce some further objects, namely

$$\boldsymbol{u}(a) = (\mathbf{1}+\boldsymbol{M})^{-1}(a\mathbf{1}+\boldsymbol{K})^{-1}\boldsymbol{r}, \tag{2.34a}$$

[7] Equation (2.29) was first established as an integrable lattice equation in [20], but was also studied in connection with discrete conformal function theory, [10]. Interestingly, this equation has also appeared in the context of numerical analysis, in connection with the Padé tables in work by R Cordellier [12].

$$ {}^t\boldsymbol{u}(b) = {}^t\boldsymbol{c}(b\mathbf{1}+\boldsymbol{K})^{-1}(\mathbf{1}+\boldsymbol{M})^{-1}, \tag{2.34b}$$

$$ S(a,b) = {}^t\boldsymbol{c}(b\mathbf{1}+\boldsymbol{K})^{-1}(\mathbf{1}+\boldsymbol{M})^{-1}(a\mathbf{1}+\boldsymbol{K})^{-1}\boldsymbol{r}. \tag{2.34c}$$

It can be shown that the latter object has the symmetry,

$$ S(a,b) = {}^t\boldsymbol{c}(b\mathbf{1}+\boldsymbol{K})^{-1}\boldsymbol{u}(a) = {}^t\boldsymbol{u}(b)(a\mathbf{1}+\boldsymbol{K})^{-1}\boldsymbol{r} = S(b,a), \tag{2.35}$$

in which $a, b \in \mathbb{C}$ are arbitrary parameters.

Following a similar derivation as that leading to (2.11) and (2.12), we can derive the following relations:

$$ (p\mathbf{1}-\boldsymbol{K})\widetilde{\boldsymbol{u}}(a) = \widetilde{V}(a)\boldsymbol{u}^{(0)} + (p-a)\boldsymbol{u}(a), \tag{2.36a}$$

$$ (p\mathbf{1}+\boldsymbol{K})\boldsymbol{u}(a) = -V(a)\widetilde{\boldsymbol{u}}^{(0)} + (p+a)\widetilde{\boldsymbol{u}}(a). \tag{2.36b}$$

Furthermore, by multiplying (2.36) from the left by the row vector ${}^t\boldsymbol{c}(b\mathbf{1}+\boldsymbol{K})^{-1}$, it can be shown that (2.36) lead to

$$ 1-(p+b)\widetilde{S}(a,b) + (p-a)S(a,b) = \widetilde{V}(a)V(b). \tag{2.37a}$$

This and its companion equation,

$$ 1-(q+b)\widehat{S}(a,b) + (q-a)S(a,b) = \widehat{V}(a)V(b), \tag{2.37b}$$

form one of the basic relations for the rest of this paper. In fact, by setting $a=-p, b=-q$ in (2.37b) we obtain

$$ 1+(p+q)S(-p,-q) = \frac{T_pT_qf}{f} = \frac{\widehat{\widetilde{f}}}{f}, \tag{2.38}$$

which allows us to express the quantity $S(p,q)$ in terms of the τ-function.

We will now use both relations (2.37a) and (2.37b), together with the symmetry $S(a,b) = S(b,a)$, cf (2.35), to deduce now a partial difference equation for $S(a,b)$ for any fixed a, b. In fact, from the identity

$$ \frac{\widehat{\widetilde{V}(a)V(b)}}{\widetilde{\widehat{V}(a)V(b)}} = \frac{\widehat{V}(b)V(a)}{\widetilde{V}(b)V(a)}, $$

by inserting (2.37) and its counterpart, with p replaced by q and the $\widetilde{\ }$-shift replaced by the $\widehat{\ }$-shift, as well as the relations with a and b interchanged, we obtain the following closed-form equation for $S(a,b)$:

$$ \frac{1-(p+b)\widehat{\widetilde{S}}(a,b)+(p-a)\widehat{S}(a,b)}{1-(q+b)\widehat{\widetilde{S}}(a,b)+(q-a)\widetilde{S}(a,b)} = \frac{1-(q+a)\widetilde{S}(a,b)+(q-b)S(a,b)}{1-(p+a)\widehat{S}(a,b)+(p-b)S(a,b)}. \tag{2.39}$$

In equation (2.39) the parameters a and b are assumed fixed, and for each choice of them we have a quadrilateral PΔE which is integrable in the sense of the multidimensional consistency property, explained in, e.g., [11, 25], where p and q play the role of lattice parameters. We will elucidate the role of the parameters a, b in subsequent sections. It suffices here to observe that by fixing the special choice $a = p, b = -p$ or $a = q, b = -q$, respectively in (2.37), we obtain the relations

$$ \widetilde{V}(p)V(-p) = 1, \qquad \widehat{V}(q)V(-q) = 1. \tag{2.40}$$

There are additional relations, connecting the object $V(a)$ with the variable w defined in (2.22), which will play an important role in the proofs of the main results. Such relations can be obtained by eliminating the terms containing the product $\boldsymbol{K}\boldsymbol{u}(a)$ in relations (2.36a) and

(2.36*b*) and their counterparts in the other lattice direction, and subsequently multiplying the resulting combinations from the left by ${}^{t}\boldsymbol{c}$, using the fact that ${}^{t}\boldsymbol{c}\boldsymbol{K}\boldsymbol{u}(a) = 1 - V(a)$, ${}^{t}\boldsymbol{c}\boldsymbol{u}_0 = w$. Thus, we arrive at the following list of relations:

$$p - q + \widehat{w} - \widetilde{w} = (p+a)\frac{\widetilde{V}(a)}{V(a)} - (q+a)\frac{\widehat{V}(a)}{V(a)} \tag{2.41a}$$

$$= (p-a)\frac{\widehat{V}(a)}{\widehat{\widetilde{V}}(a)} - (q-a)\frac{\widetilde{V}(a)}{\widehat{\widetilde{V}}(a)}, \tag{2.41b}$$

$$p + q + w - \widehat{\widetilde{w}} = (p+a)\frac{\widehat{\widetilde{V}}(a)}{\widehat{V}(a)} + (q-a)\frac{V(a)}{\widehat{V}(a)} \tag{2.41c}$$

$$= (p-a)\frac{V(a)}{\widetilde{V}(a)} + (q+a)\frac{\widehat{\widetilde{V}}(a)}{\widetilde{V}(a)}. \tag{2.41d}$$

As a direct corollary, the equality on the right-hand sides of equations (2.41*a*) and (2.41*b*), or equivalently (2.41*c*) and (2.41*d*), actually provides us with another integrable lattice equation for the variable $V(a)$. Once again, the parameter a plays a distinct role in this equation from the lattice parameters p and q, and in terms of the latter parameters the equation for $V(a)$ is multidimensionally consistent.

Remark. Note that closed-form equations for $V(a)$ are obtained by equating the right-hand sides of (2.41*a*) and (2.41*b*), or equivalently the right-hand sides of (2.41*c*) and (2.41*d*). Furthermore, breaking the covariance between the lattice directions by choosing $a = p$ in (2.41) we get for $V(p)$ the following quadrilateral PΔE:

$$2p\frac{\widetilde{V}(p)}{V(p)} = (p+q)\frac{\widehat{V}(p)}{V(p)} + (p-q)\frac{\widetilde{V}(p)}{\widehat{\widetilde{V}}(p)}. \tag{2.42}$$

Although this equation is not multidimensionally consistent in the strong sense (demanding consistency of the same equation in all lattice directions) it is multidimensionally consistent in a weaker sense (consistency between different equations on different sublattices). In fact, we can supplement (2.42) by a similar equation with a lattice variable h instead of m, and with the lattice parameter q replaced by r. These two 3-term relations are consistent around the cube with a 4-term lattice equation for $V(p)$ of the form arising from the right-hand sides of (2.41*a*), (2.41*b*) in the lattice directions associated with parameters q and r.

Using now (2.33) to substitute the variable $V(p)$, and similarly doing the same for $V(q)$ in the analogous equation obtained by interchanging p and q, and $\widetilde{\ }$-shifts and $\widehat{\ }$-shifts, we obtain the following two *bilinear* partial difference equations for the τ-function f, namely

$$(p+q)\widehat{\underset{\sim}{f}}\,\widetilde{f} + (p-q)\,\underset{\sim}{f}\,\widehat{\widetilde{f}} = 2pf\widehat{f}, \tag{2.43a}$$

$$(p+q)\widetilde{\underset{\frown}{f}}\,\widehat{f} + (q-p)\underset{\frown}{f}\,\widehat{\widetilde{f}} = 2qf\widetilde{f}. \tag{2.43b}$$

Here the under-accents $\underset{\sim}{f}$ and $\underset{\frown}{f}$ denote lattice shifts in the opposite directions to $\widetilde{f}$ and $\widehat{f}$, respectively. It can be shown that these two 6-point equations are consistent on the two-dimensional lattice from an initial-value point of view, but we will not go into this here, cf [24]. We just mention that by performing this computation explicitly, and eliminating intermediate values on vertices in the lattice one can derive from (2.43) the following 5-point lattice equation:

$$(p-q)^2 \underset{\wedge}{\underset{\sim}{f}}\,\widehat{\widetilde{f}} - (p+q)^2 \underset{\wedge}{\widetilde{f}}\,\underset{\sim}{\widehat{f}} + 4pqf^2 = 0, \tag{2.44}$$

which is Hirota's *discrete-time Toda equation*, cf [17].

2.4. Lattice KdV and lattice MKdV

We finish this section by presenting the actual lattice KdV, which is related to the lattice *potential* KdV equation (2.21) by considering differences of the variables w along the diagonals. These differences can be expressed in terms of the τ-function, namely by setting $a = p$ or $a = -p$ in equations (2.41), leading to the relations

$$\Xi \equiv p - q + \widehat{w} - \widetilde{w} = (p-q)\frac{\widehat{\widetilde{f}}\,f}{\widetilde{f}\,\widehat{f}}, \tag{2.45a}$$

$$\Upsilon \equiv p + q + w - \widehat{\widetilde{w}} = (p+q)\frac{\widetilde{f}\,\widehat{f}}{f\,\widehat{\widetilde{f}}}, \tag{2.45b}$$

from which we have immediately

$$\Xi\Upsilon = p^2 - q^2, \qquad \Xi - \widehat{\widetilde{\Xi}} = \widehat{\Upsilon} - \widetilde{\Upsilon}, \tag{2.46}$$

which leads to the lattice KdV equation, [17], in terms of either Ξ or Υ, by eliminating one or the other variable using the first relation, i.e.

$$\Xi - \widehat{\widetilde{\Xi}} = (p^2 - q^2)\left(\frac{1}{\widehat{\Xi}} - \frac{1}{\widetilde{\Xi}}\right) \quad \Leftrightarrow \quad \widehat{\Upsilon} - \widetilde{\Upsilon} = (p^2 - q^2)\left(\frac{1}{\Upsilon} - \frac{1}{\widehat{\widetilde{\Upsilon}}}\right). \tag{2.47}$$

What is not well known is that the resulting equation admits a *scalar* Lax pair of the form

$$\widehat{\widetilde{\varphi}} = \Upsilon\widehat{\varphi} + \lambda\varphi, \tag{2.48a}$$

$$\widetilde{\varphi} = \widehat{\varphi} + \Xi\varphi, \tag{2.48b}$$

where $\lambda = k^2 - q^2$ is the spectral parameter. In a similar way, from the potential lattice MKdV equation (2.26), by considering ratios of the variable v over the diagonals in the lattice, we can obtain the following lattice equation for the variable $W \equiv \widehat{v}/\widetilde{v}$:

$$\frac{\widehat{\widetilde{W}}}{W} = \frac{(p\widehat{W} - q)}{(p - q\widehat{W})}\frac{(p - q\widetilde{W})}{(p\widetilde{W} - q)}. \tag{2.49}$$

The PΔE (2.49), which we identify with the (non-potential) lattice MKdV equation, arises as the compatibility condition of the following Lax pair:

$$\widehat{\psi} = W\widetilde{\psi} + (q - pW)\,\psi, \tag{2.50a}$$

$$\widehat{\widetilde{\psi}} = \frac{p^2 - q^2}{p - qW}\widehat{\psi} + \lambda\frac{pW - q}{p - qW}\psi, \tag{2.50b}$$

where ψ is a scalar function and in which λ is the same spectral parameter. For the variable W we have the following identifications in terms of the τ-function:

$$v = V(0) = \frac{T_0 f}{f} \quad \Rightarrow \quad W = \frac{\widetilde{f}\widehat{g}}{\widehat{f}\widetilde{g}}, \tag{2.51}$$

in which $g \equiv T_0 f = \det(\mathbf{1} - \boldsymbol{M})$ is the shift in a direction with the lattice parameter $a = 0$ of the τ-function. Using equations (2.43), which also hold for g, together with relations between f, g (which follow from (2.43) taking one or the other of the lattice parameters equal to zero), we obtain

$$g\widetilde{f} + \underset{\sim}{f}\widetilde{g} = 2fg, \tag{2.52a}$$

$$\underset{\frown}{g}\widehat{f} + \underset{\frown}{f}\widehat{g} = 2fg. \tag{2.52b}$$

Finally, to obtain some explicit formulae for the τ-function (2.7) we can use the properties of the Cauchy matrix $\boldsymbol{M}$, in particular the fact that we can explicitly obtain its determinant in a factorized form. We invoke the explicit formula for Cauchy determinants

$$\det(\boldsymbol{A}) = \frac{\prod_{i<j}(k_i - k_j)(l_j - l_i)}{\prod_{i,j}(k_i - l_j)} \tag{2.53}$$

in which $\boldsymbol{A}$ is a Cauchy matrix with entries of the form

$$A_{i,j} = \frac{1}{k_i - l_j}, \qquad i, j = 1, \dots, N,$$

where the $k_i, l_j, i, j = 1, \dots, N$, are a collection of distinct parameters. Noting that the matrix $\boldsymbol{M}$ can be written as a Cauchy matrix of the form $\boldsymbol{A}$ with $l_j = -k_j$ multiplied from the left by a diagonal matrix with entries ρ_i and from the right by a diagonal matrix with entries c_j, we have

$$\det\left(\frac{\rho_i c_j}{k_i + k_j}\right) = \left(\prod_i \frac{\rho_i c_i}{2k_i}\right) \prod_{i<j}\left(\frac{k_i - k_j}{k_i + k_j}\right)^2, \tag{2.54}$$

in which i, j can run over any subset of indices of $\{1, \dots, N\}$.

Furthermore, we have the following expansion formula for the determinant of a matrix of the form $\mathbf{1} + \boldsymbol{M}$:

$$\det(\mathbf{1} + \boldsymbol{M}) = 1 + \sum_{i=1}^{N} |M_{i,i}| + \sum_{i<j} \begin{vmatrix} M_{i,i} & M_{i,j} \\ M_{j,i} & M_{j,j} \end{vmatrix} + \sum_{i<j<k} \begin{vmatrix} M_{i,i} & M_{i,j} & M_{i,k} \\ M_{j,i} & M_{j,j} & M_{j,k} \\ M_{k,i} & M_{k,j} & M_{k,k} \end{vmatrix} + \cdots + \det(\boldsymbol{M}), \tag{2.55}$$

where as a consequence of (2.54) we can compute all the terms in the expansion (2.55) explicitly, leading to a form similar to the celebrated N-soliton form for the KdV equation given by Hirota [17].

3. From lattice KdV soliton solutions to Q3 solitons

The main lattice equation arising from the scheme setup in section 2, i.e. comprising the lattice equations that we coined to be of KdV type, is equation (2.39) which to our knowledge was first given in [23], cf also [28], and which we can rewrite in an affine linear form as follows:

$$[1 + (p-a)S - (p+b)\widetilde{S}][1 + (p-b)\widehat{S} - (p+a)\widehat{\widetilde{S}}] = [1 + (q-a)S - (q+b)\widehat{S}][1 + (q-b)\widetilde{S} - (q+a)\widehat{\widetilde{S}}]. \tag{3.1}$$

Following [29], we will refer to (3.1) as the NQC equation. We begin this section by making precise the connection between the NQC equation and the equation $Q3_{\delta=0}$. This provides some (mainly notational) ground work for the subsequent statement and proof of the *N*-soliton solution for the equation Q3, which is constructed on the basis of solutions of (3.1). In [29], a full classification of all parameter subcases of (3.1) was given, In particular, it was already noted in the earlier papers that by limits on the parameters a, b (as either tend to zero or infinity) equation (3.1) reduces to the other KdV-type lattice equations, namely (2.21), (2.26) as well as (2.29). Furthermore, as already remarked in [3], (3.1) corresponds to the $\delta = 0$ case of Q3. We will first make this connection more explicit, leading to a new parametrization of Q3, and then show that, in fact, the *N*-soliton solutions of (3.1) for different values of a, b together constitute a solution for the full case (i.e. $\delta \neq 0$) of Q3, which is the main statement in theorem 1.

3.1. Connection between $Q3_{\delta=0}$ *and the NQC equation*

We shall now relate equation (3.1) to the special case of (1.1*c*) with $\delta = 0$, which we indicate by $Q3_{\delta=0}$. We first introduce the following dependent variable:

$$u^0_{n,m} = \digamma_{n,m}(a,b)(1-(a+b)S_{n,m}(a,b)), \tag{3.2}$$

in which

$$\digamma_{n,m}(a,b) = \left(\frac{P}{(p-a)(p-b)}\right)^n \left(\frac{Q}{(q-a)(q-b)}\right)^m, \tag{3.3}$$

where

$$P^2 = (p^2-a^2)(p^2-b^2), \qquad Q^2 = (q^2-a^2)(q^2-b^2). \tag{3.4}$$

This brings (3.1) in the form

$$P(u^0\widehat{u}^0 + \widetilde{u}^0\widehat{\widetilde{u}}^0) - Q(u^0\widetilde{u}^0 + \widehat{u}^0\widehat{\widetilde{u}}^0) = (p^2-q^2)(\widehat{u}^0\widetilde{u}^0 + u^0\widehat{\widetilde{u}}^0), \tag{3.5}$$

where the lattice parameters have now become points $\mathfrak{p} = (p, P)$, respectively, $\mathfrak{q} = (q, Q)$ on the (Jacobi) elliptic curve:

$$\mathfrak{p}, \mathfrak{q} \in \Gamma := \{(x, X) | X^2 = (x^2-a^2)(x^2-b^2)\}. \tag{3.6}$$

The full equation Q3 now reads

$$P(u\widehat{u} + \widetilde{u}\widehat{\widetilde{u}}) - Q(u\widetilde{u} + \widehat{u}\widehat{\widetilde{u}}) = (p^2-q^2)\left((\widehat{u}\widetilde{u} + u\widehat{\widetilde{u}}) + \frac{\delta^2}{4PQ}\right), \tag{3.7}$$

and this corresponds to the form of the Q3 equation (1.1*c*) in the ABS list by the following relations between the original parameters $\mathring{p}, \mathring{q}$ and the new parameters $\mathfrak{p}, \mathfrak{q}$:

$$\mathring{p}^2 = \frac{p^2-b^2}{p^2-a^2}, \qquad P = \frac{(b^2-a^2)\mathring{p}}{1-\mathring{p}^2}, \qquad \mathring{q}^2 = \frac{q^2-b^2}{q^2-a^2}, \qquad Q = \frac{(b^2-a^2)\mathring{q}}{1-\mathring{q}^2}, \tag{3.8}$$

while $u = (b^2-a^2)\mathring{u}$, the latter being the dependent variable of equation (1.1*c*).

3.2. N-soliton structure for Q3

The main result of the paper [7] was to give the *N*-soliton solution for the equation Q3, however, without presenting there the full proof. Here we will present a constructive proof based on the machinery developed in section 2, which we think reveals some of the structures behind the Q3 equation and its solutions, and their connection to other lattice equations.

Theorem 1. *The N-soliton solution of* Q3 *(3.7), which we denote by* $u^{(N)} = u^{(N)}_{n,m}$, *is given by the formula*

$$u^{(N)} = A\digamma(a,b)[1-(a+b)S(a,b)] + B\digamma(a,-b)[1-(a-b)S(a,-b)] \\ + C\digamma(-a,b)[1+(a-b)S(-a,b)] + D\digamma(-a,-b)[1+(a+b)S(-a,-b)]. \tag{3.9}$$

Here $S(\pm a, \pm b) = S_{n,m}(\pm a, \pm b)$ *are the N-soliton solutions of the NQC equation (3.1) with parameters* $\pm a, \pm b$ *as given in (2.35). The function* $\digamma(a,b) = \digamma_{n,m}(a,b)$ *is defined in (3.3), and* A, B, C *and* D *are constants subject to the single constraint:*

$$AD(a+b)^2 - BC(a-b)^2 = -\frac{\delta^2}{16ab}. \tag{3.10}$$

Remark. The pseudo-linear structure of this solution, as an almost arbitrary linear combination of four different solutions of the NQC equation with the parameters a, b changing signs, is remarkable, the various choices of signs apparently being connected to various choices of pairs of branch points of the elliptic curve (3.4) of the lattice parameters. As we shall see in the unravelling of this solution, the a, b not only play the role of moduli of those curves, but as lattice parameters in their own right (w.r.t. additional 'hidden' lattice directions).

Proof. We will now go over the various steps needed to prove theorem 1, all of which are based on the relations of the N-soliton solutions to KdV-type equations given in section 2.

Step # 1. We first introduce a new associated dependent variable $U^{(N)} = U^{(N)}_{n,m}$ given by

$$U^{(N)} = (a+b)A\digamma(a,b)V(a)V(b) + (a-b)B\digamma(a,-b)V(a)V(-b) \\ -(a-b)C\digamma(-a,b)V(-a)V(b) - (a+b)D\digamma(-a,-b)V(-a)V(-b), \tag{3.11}$$

in which $V(\pm a) = V_{n,m}(\pm a)$, $V(\pm b) = V_{n,m}(\pm b)$ are defined by (2.32) and $\digamma(a,b) = \digamma_{n,m}(a,b)$ is defined by (3.3). We recall that the connection between the objects $V(a)$, $V(b)$ and $S(a,b)$ is given by (2.37), and that these relations can be covariantly extended to other lattice directions. Using these with equations (2.41) we can derive the following set of important Miura-type relations.

Lemma 1. *For* $u = u^{(N)}_{n,m}$ *defined in (3.9) and the associated variable* $U = U^{(N)}_{n,m}$ *defined in (3.11) the following hold:*

$$p - q + \widehat{w} - \widetilde{w} = \frac{1}{U}[P\widetilde{u} - Q\widehat{u} - (p^2-q^2)u] \tag{3.12a}$$

$$= -\frac{1}{\widehat{\widetilde{U}}}[P\widehat{u} - Q\widetilde{u} - (p^2-q^2)\widehat{\widetilde{u}}] \tag{3.12b}$$

$$p + q + w - \widehat{\widetilde{w}} = \frac{1}{\widehat{U}}[P\widehat{\widetilde{u}} - Qu - (p^2-q^2)\widehat{u}] \tag{3.12c}$$

$$= -\frac{1}{\widetilde{U}}[Pu - Q\widehat{\widetilde{u}} - (p^2-q^2)\widetilde{u}], \tag{3.12d}$$

where $w = w^{(N)}_{n,m}$, *defined in (2.22), is the N-soliton solution of the lattice potential KdV equation (2.21). Relations (3.12) hold for arbitrary coefficients* A, B, C, D *(i.e., without invoking the constraint (3.10)).*

Proof. This is by direct computation. In fact, to prove (3.12*a*), we consider

$$
\begin{aligned}
P\widetilde{u} - Q\widehat{u} - (p^2-q^2)u &= A\digamma(a,b)[(p+a)(p+b)(1-(a+b)\widetilde{S}(a,b)) \\
&\quad - (q+a)(q+b)(1-(a+b)\widehat{S}(a,b) - (p^2-q^2)(1-(a+b)S(a,b))] + \cdots \\
&= A(a+b)\digamma(a,b)[(p-q) - (p+a)(p+b)\widetilde{S}(a,b) \\
&\quad + (q+a)(q+b)\widehat{S}(a,b) + (p^2-q^2)S(a,b)] + \cdots \\
&= A(a+b)\digamma(a,b)[(p+a)(1-(p+b)\widetilde{S}(a,b)) + (p-a)S(a,b)) \\
&\quad - (q+a)(1-(q+b)\widehat{S}(a,b) + (q-a)S(a,b))] + \cdots \\
&= A(a+b)\digamma(a,b)[(p+a)\widetilde{V}(a)V(b) - (q+a)\widehat{V}(a)V(b)] + \cdots \\
&= A(a+b)\digamma(a,b)(p-q+\widehat{w}-\widetilde{w})V(a)V(b) + \cdots \\
&= (p-q+\widehat{w}-\widetilde{w})U,
\end{aligned}
$$

in which the dots in each line on the right-hand sides stand for similar terms with (a, b) replaced by $(a, -b)$, $(-a, b)$, $(-a, -b)$ and A replaced by B, C, D, respectively. In the last steps we have made use of (2.37) and (2.41*a*), respectively. The other relations (3.12*b*)–(3.12*d*) are proven in a similar fashion. □

Step # 2. We now establish an important property of the objects defined in (3.9) and (3.11), which holds for arbitrary coefficients A, B, C, D (i.e., again without invoking the constraint (3.10)), namely,

Lemma 2. *The following identities hold for the N-soliton expressions,* $u = u_{n,m}^{(N)}$, $U = U_{n,m}^{(N)}$*:*

$$U\widetilde{U} - P(u^2+\widetilde{u}^2) + (2p^2-a^2-b^2)u\widetilde{u} = \frac{4ab}{P}\det(\mathcal{A}), \tag{3.13a}$$

$$U\widehat{U} - Q(u^2+\widehat{u}^2) + (2q^2-a^2-b^2)u\widehat{u} = \frac{4ab}{Q}\det(\mathcal{A}), \tag{3.13b}$$

in which the 2×2 *matrix* $\mathcal{A}$ *is given by*

$$\mathcal{A} = \begin{pmatrix} (a+b)A & (a-b)B \\ -(a-b)C & -(a+b)D \end{pmatrix}. \tag{3.14}$$

Proof. Again this is proven by direct computation. However, in order to make the emergence of the determinants more transparent we consider the following 2×2 matrices:

$$\boldsymbol{L} =: \begin{pmatrix} P\widetilde{u} - (p^2-b^2)u, & (p-b)U \\ (p+b)\widetilde{U}, & -Pu + (p^2-b^2)\widetilde{u} \end{pmatrix}, \tag{3.15a}$$

$$\boldsymbol{M} =: \begin{pmatrix} Q\widehat{u} - (q^2-b^2)u, & (q-b)U \\ (q+b)\widehat{U}, & -Qu + (q^2-b^2)\widehat{u} \end{pmatrix}. \tag{3.15b}$$

Evaluating the entries in these matrices, we obtain

$$
\begin{aligned}
&P\widetilde{u} - (p^2-b^2)u \\
&= A\digamma(a,b)\left[\frac{P^2}{(p-a)(p-b)}(1-(a+b)\widetilde{S}(a,b)) - (p^2-b^2)(1-(a+b)S(a,b))\right] + \cdots \\
&= A\digamma(a,b)(p+b)[(p+a)(1-(a+b)\widetilde{S}(a,b)) - (p-b)(1-(a+b)S(a,b))] + \cdots \\
&= A\digamma(a,b)(p+b)(a+b)[1-(p+a)\widetilde{S}(a,b)) + (p-b)S(a,b)] + \cdots \\
&= A(a+b)\digamma(a,b)(p+b)\widetilde{V}(b)V(a) + \cdots = \sqrt{p^2-b^2}\,\boldsymbol{r}^T(a)\mathcal{A}\widetilde{\boldsymbol{r}}(b),
\end{aligned} \tag{3.16}
$$

in which again the dots in each line on the right-hand sides stand for similar terms with (a, b) replaced by $(a, -b)$, $(-a, b)$, $(-a, -b)$ and A replaced by B, C, D, respectively. On the right-hand side of this expression, we have introduced the vectors

$$\boldsymbol{r}(a) = \begin{pmatrix} \rho^{1/2}(a)V(a) \\ \rho^{1/2}(-a)V(-a) \end{pmatrix}, \qquad \boldsymbol{r}^T(b) = (\rho^{1/2}(b)V(b), \rho^{1/2}(-b)V(-b)),$$

in which the plane-wave factors $\rho(a)$ are given by

$$\rho(a) = \left(\frac{p+a}{p-a}\right)^n \left(\frac{q+a}{q-a}\right)^m. \tag{3.17}$$

A similar computation as above yields

$$-Pu + (p^2 - b^2)\widetilde{u} = \sqrt{p^2 - b^2}\,\widetilde{\boldsymbol{r}}^T(a)\mathcal{A}\boldsymbol{r}(a), \tag{3.18}$$

whilst U and $\widetilde{U}$ can be written as

$$U = \boldsymbol{r}^T(a)\mathcal{A}\boldsymbol{r}(b), \qquad \widetilde{U} = \widetilde{\boldsymbol{r}}^T(a)\mathcal{A}\widetilde{\boldsymbol{r}}(b). \tag{3.19}$$

Thus, we find that the matrix $\boldsymbol{L}$ in (3.15) can be written as

$$\boldsymbol{L} = \begin{pmatrix} \sqrt{p^2 - b^2}\,\boldsymbol{r}^T(a)\mathcal{A}\widetilde{\boldsymbol{r}}(b), & (p-b)\boldsymbol{r}^T(a)\mathcal{A}\boldsymbol{r}(b) \\ (p+b)\widetilde{\boldsymbol{r}}^T(a)\mathcal{A}\widetilde{\boldsymbol{r}}(b), & \sqrt{p^2 - b^2}\,\widetilde{\boldsymbol{r}}^T(a)\mathcal{A}\boldsymbol{r}(b) \end{pmatrix},$$

and similarly for the matrix $\boldsymbol{M}$. Using now the general determinantal identity,

$$\det\left(\sum_{j=1}^{r} \boldsymbol{x}_j \boldsymbol{y}_j^T\right) = \det\left(\left(\boldsymbol{y}_i^T \cdot \boldsymbol{x}_j\right)_{i,j=1,\dots,r}\right),$$

for any collection of r pairs of r-component column vectors $\boldsymbol{x}_i$, $\boldsymbol{y}_i$ (the superindex T denoting transposition), we obtain the following result:

$$\begin{aligned}
\det(\boldsymbol{L}) &= (p^2 - b^2)\det(\mathcal{A}\widetilde{\boldsymbol{r}}(b)\boldsymbol{r}^T(a) + \mathcal{A}\boldsymbol{r}(b)\widetilde{\boldsymbol{r}}^T(a)) \\
&= (p^2 - b^2)\det(\mathcal{A})\det(\widetilde{\boldsymbol{r}}(b)\boldsymbol{r}^T(a) + \boldsymbol{r}(b)\widetilde{\boldsymbol{r}}^T(a)) \\
&= (p^2 - b^2)\det(\mathcal{A})\det\left\{(\widetilde{\boldsymbol{r}}(b), \boldsymbol{r}(b))\begin{pmatrix} \boldsymbol{r}^T(a) \\ \widetilde{\boldsymbol{r}}^T(a) \end{pmatrix}\right\} \\
&= -(p^2 - b^2)\det(\mathcal{A})\det(\boldsymbol{r}(a), \widetilde{\boldsymbol{r}}(a))\det(\boldsymbol{r}(b), \widetilde{\boldsymbol{r}}(b)).
\end{aligned}$$

It remains to compute the determinant of the matrix $(\boldsymbol{r}(a), \widetilde{\boldsymbol{r}}(a))$ whose columns are the 2-component vectors $\boldsymbol{r}(a)$ and $\widetilde{\boldsymbol{r}}(a)$. This is done as follows:

$$\begin{aligned}
\det(\boldsymbol{r}(a), \widetilde{\boldsymbol{r}}(a)) &= \rho^{1/2}(a)\widetilde{\rho}^{1/2}(-a)V(a)\widetilde{V}(-a) - \widetilde{\rho}^{1/2}(a)\rho^{1/2}(-a)\widetilde{V}(a)V(-a) \\
&= \sqrt{\frac{p-a}{p+a}}V(a)\widetilde{V}(-a) - \sqrt{\frac{p+a}{p-a}}\widetilde{V}(a)V(-a) \\
&= \sqrt{\frac{p-a}{p+a}}[1 - (p+a)\widetilde{S}(a,-a) + (p+a)S(a,-a)] \\
&\quad - \sqrt{\frac{p+a}{p-a}}[1 - (p-a)\widetilde{S}(-a,a) + (p-a)S(-a,a)] \\
&= -\frac{2a}{\sqrt{p^2 - a^2}},
\end{aligned}$$

where we have used (2.37) for the choices $b = -a$ and the signs of a, b reversed, as well as the fact that $S(a, b) = S(b, a)$. Thus, putting everything together we obtain the result

$$\det(\boldsymbol{L}) = -(p^2 - b^2)\det(\mathcal{A})\frac{4ab}{\sqrt{p^2 - a^2}\sqrt{p^2 - b^2}}. \tag{3.20}$$

On the other hand, a direct computation of the determinant gives

$$\begin{aligned}\det(\boldsymbol{L}) &= -[P\widetilde{u} - (p^2-b^2)u][Pu - (p^2-b^2)\widetilde{u}] - (p^2-b^2)U\widetilde{U}\\ &= (p^2-b^2)[P(u^2+\widetilde{u}^2) - (2p^2-a^2-b^2)u\widetilde{u} - U\widetilde{U}]. \end{aligned}\tag{3.21}$$

Comparing the two expressions for $\det(\boldsymbol{L})$ from (3.20) and (3.21) we obtain the first equation in lemma 2. □

Step # 3. The last step is by combining relations (3.12) and (3.13) as well as the potential lattice KdV equation (2.21) to assert that $u = u_{n,m}^{(N)}$ solves the Q3 equation. In fact, multiplying, for instance, (3.12*b*) by (3.12*d*) and using (3.13*b*), where we identify

$$\det(\mathcal{A}) = \frac{\delta^2}{16ab}$$

according to (3.10), we obtain from the lattice potential KdV:

$$\begin{aligned}p^2-q^2 &= (p+q+w-\widehat{\widetilde{w}})(p-q+\widehat{w}-\widetilde{w})\\
&= \frac{1}{\widetilde{U}\widehat{U}}[P\widehat{u} - Q\widetilde{u} - (p^2-q^2)\widehat{\widetilde{u}}][Pu - Q\widehat{\widetilde{u}} - (p^2-q^2)\widetilde{u}]\\
\Rightarrow&\quad (p^2-q^2)\left[Q(\widetilde{u}^2+\widehat{u}^2) - (2q^2-a^2-b^2)\widetilde{u}\widehat{u} + \frac{\delta^2}{4Q}\right]\\
&= P^2(u\widehat{u}) + Q^2(\widetilde{u}\widehat{\widetilde{u}}) + (p^2-q^2)^2\widetilde{u}\widehat{u} - PQ(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}})\\
&\quad - (p^2-q^2)P(\widehat{u}\widetilde{u} + u\widehat{\widetilde{u}}) + (p^2-q^2)Q(\widetilde{u}^2+\widehat{\widetilde{u}}^2)\\
&= P^2(u\widehat{u} + \widetilde{u}\widehat{\widetilde{u}}) + (Q^2-P^2)(\widetilde{u}\widehat{\widetilde{u}}) + (p^2-q^2)^2\widetilde{u}\widehat{u} - PQ(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}})\\
&\quad - (p^2-q^2)P(\widehat{u}\widetilde{u} + u\widehat{\widetilde{u}}) + (p^2-q^2)Q(\widetilde{u}^2+\widehat{\widetilde{u}}^2)\\
\Rightarrow&\quad (p^2-q^2)\left[-(2q^2-a^2-b^2)\widetilde{u}\widehat{u} + \frac{\delta^2}{4Q}\right]\\
&= P[P(u\widehat{u} + \widetilde{u}\widehat{\widetilde{u}}) - Q(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}}) - (p^2-q^2)(\widehat{u}\widetilde{u} + u\widehat{\widetilde{u}})]\\
&\quad + (p^2-q^2)[(a^2+b^2-p^2-q^2)\widetilde{u}\widehat{u} + (p^2-q^2)u\widehat{\widetilde{u}}],\end{aligned}$$

where we have used the fact that $\mathfrak{p}, \mathfrak{q}$ are on the elliptic curve (3.4) and the identification of $\widetilde{U}\widehat{\widetilde{U}}$ via the result (3.13) of lemma 2. From the last step, after some cancellation of terms, we obtain Q3, in the form (3.7), for the function $u = u_{n,m}^{(N)}$, which completes the proof of the theorem. □

Thus, we have established the explicit form of the N-soliton solution (3.9) of the full Q3 equation in the form (3.7) in a constructive way, using basically all the ingredients of the lattice KdV-type soliton structures of section 2. The parametrization using the KdV lattice parameters p, q, in terms of which the underlying plane-wave factors (2.2) take a simple form, now involving some new fixed parameters a, b, in terms of which we have an elliptic curve (3.4) appearing in the equation, of which $\pm a, \pm b$ are the branch points. It turns out that it is natural now to covariantly extend the lattice to include lattice directions for which these parameters a, b are the lattice parameters, and thus to introduce extensions of the plane-wave factors of the form [7]

$$\rho_{n,m,\alpha,\beta}(k_i) := \left(\frac{p+k_i}{p-k_i}\right)^n \left(\frac{q+k_i}{q-k_i}\right)^m \left(\frac{a+k_i}{a-k_i}\right)^\alpha \left(\frac{b+k_i}{b-k_i}\right)^\beta \rho_i^0. \tag{3.22}$$

This allows us then to use relations of the types (2.33) and (2.38) to express the objects $V(a)$ and $S(a,b)$ in terms of a τ-function, $f = f_{n,m,\alpha,\beta}$, as follows:

$$V(a) = \frac{T_{-a}f}{f} = \frac{T_a^{-1}f}{f}, \qquad 1-(a+b)S(a,b) = \frac{T_{-a}T_{-b}f}{f} = \frac{T_a^{-1}T_b^{-1}f}{f}, \tag{3.23}$$

in which T_a, T_b denote elementary shifts in the four-dimensional lattice in directions associated with the parameters a, b (i.e. shifts by one unit in the corresponding lattice variables α, β, respectively). Thus, as a consequence of theorem 1, we get the following.

Corollary. *Solutions (3.9) take on the following form in terms of the τ-function $f = f_{n,m,\alpha,\beta}$, when the plane-wave factors ρ are covariantly extended according to formula (3.22), namely*

$$\begin{aligned} u^{(N)} = {} & AF(a,b)\frac{f_{n,m,\alpha-1,\beta-1}}{f_{n,m,\alpha,\beta}} + BF(a,-b)\frac{f_{n,m,\alpha-1,\beta+1}}{f_{n,m,\alpha,\beta}} \\ & + CF(-a,b)\frac{f_{n,m,\alpha+1,\beta-1}}{f_{n,m,\alpha,\beta}} + DF(-a,-b)\frac{f_{n,m,\alpha+1,\beta+1}}{f_{n,m,\alpha,\beta}}, \end{aligned} \tag{3.24}$$

cf [7], whilst its adjoint function takes the following form:

$$\begin{aligned} U^{(N)} = {} & A(a+b)F(a,b)\frac{f_{n,m,\alpha-1,\beta}f_{n,m,\alpha,\beta-1}}{f^2_{n,m,\alpha,\beta}} + B(a-b)F(a,-b)\frac{f_{n,m,\alpha-1,\beta}f_{n,m,\alpha,\beta+1}}{f^2_{n,m,\alpha,\beta}} \\ & - C(a-b)F(-a,b)\frac{f_{n,m,\alpha+1,\beta}f_{n,m,\alpha,\beta-1}}{f^2_{n,m,\alpha,\beta}} \\ & - D(a+b)F(-a,-b)\frac{f_{n,m,\alpha+1,\beta}f_{n,m,\alpha,\beta+1}}{f^2_{n,m,\alpha,\beta}}. \end{aligned} \tag{3.25}$$

As was indicated in [7], the verification of the solution of Q3 in terms of f can also be done by establishing a set of interlinked Hirota–Miwa equations in the four-dimensional space of the variables n, m, α, β. An alternative approach to the N-soliton solutions for ABS-type equations is developed in a subsequent paper, [16], in which a formalism is developed in terms of Casorati determinant expressions for the τ-functions.

3.3. Associated biquadratic polynomials and Lax pair

In this subsection, we make some additional observations and connections related to the N-soliton solution for Q3 presented above in theorem 1. To this end it will be useful to introduce the following:

$$\mathcal{Q}_{\mathfrak{p},\mathfrak{q}}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}}) := P(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}}) - Q(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}}) - (p^2-q^2)\left((\widehat{u}\widetilde{u}+u\widehat{\widetilde{u}}) + \frac{\delta^2}{4PQ}\right). \tag{3.26}$$

This quadrilateral expression is a polynomial of degree one in four variables with coefficients which depend upon the lattice parameters $\mathfrak{p}, \mathfrak{q} \in \Gamma$ (3.6). By introducing this polynomial, equation (3.7) may be written conveniently as $\mathcal{Q}_{\mathfrak{p},\mathfrak{q}}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}}) = 0$. Following the approach of ABS [3] we may associate with this polynomial a biquadratic expression which we denote $\mathcal{H}$:

$$\mathcal{H}_{\mathfrak{p}}(u,\widetilde{u}) := P(u^2+\widetilde{u}^2) - (2p^2-a^2-b^2)u\widetilde{u} + \frac{\delta^2}{4P}. \tag{3.27}$$

This biquadratic is related to (3.26) in two ways: first, by the equations

$$\begin{aligned} (p^2-q^2)\mathcal{H}_{\mathfrak{p}}(u,\widetilde{u}) &= \mathcal{Q}_{\mathfrak{p},\mathfrak{q}}(\mathcal{Q}_{\mathfrak{p},\mathfrak{q}})_{\widehat{u}\widehat{\widetilde{u}}} - (\mathcal{Q}_{\mathfrak{p},\mathfrak{q}})_{\widehat{u}}(\mathcal{Q}_{\mathfrak{p},\mathfrak{q}})_{\widehat{\widetilde{u}}}, \\ (q^2-p^2)\mathcal{H}_{\mathfrak{q}}(u,\widehat{u}) &= \mathcal{Q}_{\mathfrak{p},\mathfrak{q}}(\mathcal{Q}_{\mathfrak{p},\mathfrak{q}})_{\widetilde{u}\widehat{\widetilde{u}}} - (\mathcal{Q}_{\mathfrak{p},\mathfrak{q}})_{\widetilde{u}}(\mathcal{Q}_{\mathfrak{p},\mathfrak{q}})_{\widehat{\widetilde{u}}}, \end{aligned} \tag{3.28}$$

where the subscripts $\widetilde{u}$, etc, denote partial derivatives and, second, by the equation

$$\mathcal{H}_{\mathfrak{p}}(u,\widetilde{u})\mathcal{H}_{\mathfrak{p}}(\widehat{u},\widehat{\widetilde{u}}) - \mathcal{H}_{\mathfrak{q}}(u,\widehat{u})\mathcal{H}_{\mathfrak{q}}(\widetilde{u},\widehat{\widetilde{u}}) = \mathcal{Q}_{\mathfrak{p},\mathfrak{q}}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}})\mathcal{Q}^*_{\mathfrak{p},\mathfrak{q}}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}}), \tag{3.29}$$

where $\mathcal{Q}^*_{\mathfrak{p},\mathfrak{q}}$ denotes an associated quadrilateral given by

$$\mathcal{Q}^*_{\mathfrak{p},\mathfrak{q}}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}}) := P(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}}) + Q(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}}) - (p^2+q^2-a^2-b^2)\left((\widehat{u}\widetilde{u}+u\widehat{\widetilde{u}}) - \frac{\delta^2}{4PQ}\right). \tag{3.30}$$

Furthermore, the underlying curve associated with (3.26) is determined by the polynomial in a single variable:

$$r(u) = (\mathcal{H}_{\mathfrak{p}})^2_{\widetilde{u}} - 2\mathcal{H}_{\mathfrak{p}}(\mathcal{H}_{\mathfrak{p}})_{\widetilde{u}\widetilde{u}} = (a^2-b^2)^2u^2 - \delta^2. \tag{3.31}$$

The proof in subsection 3.2 relies on relations for a number of quantities that have appeared in passing, but which we will now explain somewhat further. In fact, the identities (3.13) constitute factorization properties for the aforementioned biquadratics, and can be written as[8]

$$\mathcal{H}_{\mathfrak{p}}(u,\widetilde{u}) = U\widetilde{U}, \qquad \mathcal{H}_{\mathfrak{q}}(u,\widehat{u}) = U\widehat{U}. \tag{3.32}$$

As a consequence of the factorization (3.32) and the identity (3.29) it becomes clear that either the lattice equation $\mathcal{Q}_{\mathfrak{p},\mathfrak{q}}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}}) = 0$ is satisfied, or the equation $\mathcal{Q}^*_{\mathfrak{p},\mathfrak{q}}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}}) = 0$ is.

The matrices $\boldsymbol{L}$ and $\boldsymbol{M}$ appearing in the proof of lemma 2, cf (3.15), and whose determinants coincide with the biquadratics given above, are related to the Lax representation for Q3, which can be written as

$$\widetilde{\phi} = \boldsymbol{L}_{\mathfrak{k}}(u,\widetilde{u})\phi = \gamma_{\mathfrak{p}}\begin{pmatrix} P\widetilde{u}-(p^2-k^2)u, & -K \\ Ku\widetilde{u}+(p^2-k^2)\frac{\delta^2}{4PK}, & -Pu+(p^2-k^2)\widetilde{u}\end{pmatrix}\phi, \tag{3.33a}$$

$$\widehat{\phi} = \boldsymbol{M}_{\mathfrak{k}}(u,\widehat{u})\phi = \gamma_{\mathfrak{q}}\begin{pmatrix} Q\widehat{u}-(q^2-k^2)u, & -K \\ Ku\widehat{u}+(q^2-k^2)\frac{\delta^2}{4QK}, & -Qu+(q^2-k^2)\widehat{u}\end{pmatrix}\phi, \tag{3.33b}$$

in which we can set $\gamma_{\mathfrak{p}} = \mathcal{H}_{\mathfrak{p}}(u,\widetilde{u})^{-1/2}$, $\gamma_{\mathfrak{q}} = \mathcal{H}_{\mathfrak{q}}(u,\widehat{u})^{-1/2}$. In [11, 19], it was explained how to obtain, in general, Lax representations for quadrilateral lattice equations which satisfy the multidimensional consistency property. In the case of the soliton solutions, choosing u as in (3.9), and using the corresponding factorization of the biquadratics as in (3.32), with U given by (3.11), one can explicitly identify the eigenvector ϕ in terms of the ingredients of theorem 1 and the objects of section 2. In fact, the construction of the Lax representation implies that in terms of the components of the vector $\phi = (\phi_1,\phi_2)^T$ we have that the ratio $\phi_2/\phi_1 = T_k u$, where it is understood that all the variables are covariantly extended to a lattice involving a lattice direction associated with the spectral parameter k as the lattice parameter. We will not pursue this identification in the present paper.

4. Connection to the Bäcklund transformation

In this section, we study the recursive structure of the N-soliton solutions for Q3 presented in theorem 1, under the application of the relevant Bäcklund transformation. Thus, in theorem 2, we give the precise correspondence between the N- and $N+1$-soliton solutions, which, as we will see, will involve some redefinitions of the constants as we increase the soliton number. In particular, this is once again a manifestation of the multidimensional consistency of the lattice equation, by which we can interpret in a precise sense the lattice equation as its own BT. Furthermore, we highlight at the end of this section how the initial step of the Bäcklund chain, i.e. the *seed solution* defined as a fixed point of the BT, is embedded in the general formula of theorem 1.

[8] Incidentally, by performing a natural continuum limit on one of the lattice variables, one can identify the object U with a derivative of u with respect to the emerging continuum variable.

4.1. Bäcklund transform from N- to N + 1-soliton solutions

The following theorem relates the N-soliton solution given previously (3.9) to the solution one would find by the iterative application of the Bäcklund transformation.

Theorem 2. *Let $u^{(N)}$ be as defined in (3.9) and let $\overline{u}^{(N+1)}$ be equal to $u^{(N+1)}$ as defined in (3.9), depending on additional parameters k_{N+1}, c_{N+1} and additional plane-wave factor ρ_{N+1}, and where all but the latter plane-wave factors $\rho_i, i = 1, \ldots, N$, as well as the discrete exponentials $\digamma(\pm a, \pm b)$ are replaced by*

$$\overline{\rho}_i = \frac{k_{N+1} + k_i}{k_{N+1} - k_i}\rho_i, \quad (i \neq N+1), \quad \overline{\digamma}(\pm a, \pm b) = \frac{K_{N+1}}{(k_{N+1} \mp a)(k_{N+1} \mp b)}\digamma(\pm a, \pm b), \tag{4.1}$$

respectively. Then $u^{(N)}$ is related to $\overline{u}^{(N+1)}$ by the Bäcklund transformation with the Bäcklund parameter $\mathfrak{k}_{N+1} \in \Gamma$, that is the following equations hold:

$$\mathcal{Q}_{\mathfrak{p},\mathfrak{k}_{N+1}}(u^{(N)}, \widetilde{u}^{(N)}, \overline{u}^{(N+1)}, \widetilde{\overline{u}}^{(N+1)}) = 0, \qquad \mathcal{Q}_{\mathfrak{q},\mathfrak{k}_{N+1}}(u^{(N)}, \widehat{u}^{(N)}, \overline{u}^{(N+1)}, \widehat{\overline{u}}^{(N+1)}) = 0, \tag{4.2}$$

in which $\mathcal{Q}_{\mathfrak{p},\mathfrak{k}_{N+1}}, \mathcal{Q}_{\mathfrak{q},\mathfrak{k}_{N+1}}$ are the quadrilateral expressions of the form given in (3.26).

Remark. The shift $\rho_i \to \overline{\rho}_i$ corresponds to the covariant extension of the lattice variables in a direction given by the new parameter k_{N+1}, which equivalently can be described by the introduction of a new discrete lattice variable h associated with k_{N+1} as a lattice parameter. This dependence can also be incorporated in the parameters c_i, as a hidden dependence on this variable. Remarkably, the coefficients A, B, C, D, which, in principle, could alter in the transition from the N- to the $N+1$-soliton solution, in fact, remain unaltered in the incrementation of the soliton number.

Proof. The proof of theorem 2 can be broken down into a number of steps, which are all constructive, and rely once again on the machinery developed in section 2. In the first step, we break down the $N+1$-soliton expression into components associated with the N-soliton solution. In the second step, we apply the BT to the N-soliton solution and in the final step we compare the expressions obtained in the two previous steps.

Step # 1. We first establish a recursive structure between the basic objects like $S_{n,m}(a, b)$ and $V_{n,m}(a, b)$ between the N- and $N+1$-soliton solutions. This uses the breakdown of the Cauchy matrix as it occurs as the kernel $(\mathbf{1} + \boldsymbol{M})^{-1}$ in the various objects.

Lemma 3. *The following identity holds for the inverse of a $(N+1) \times (N+1)$ block matrix:*

$$\left(\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{b} \\ \hline \boldsymbol{c}^T & d \end{array}\right)^{-1} = \left(\begin{array}{c|c} \boldsymbol{A}^{-1}(\mathbf{1} + \frac{1}{s}\boldsymbol{b}\boldsymbol{c}^T\boldsymbol{A}^{-1}) & -\frac{1}{s}\boldsymbol{A}^{-1}\boldsymbol{b} \\ \hline -\frac{1}{s}\boldsymbol{c}^T\boldsymbol{A}^{-1} & \frac{1}{s} \end{array}\right),$$

in which $\boldsymbol{A}$ is an invertible $N \times N$ matrix, $\mathbf{b}$ and $\mathbf{c}^T$ are N-component column and row vectors, respectively , and d is a nonzero scalar, where the scalar quantity s, given by

$$s = d - \boldsymbol{c}^T\boldsymbol{A}^{-1}\boldsymbol{b},$$

is assumed to be nonzero.

Proof. The formula can be verified by direct multiplication. The matrix is invertible if s is nonzero. □

Let $\boldsymbol{M}^{(N+1)}$ be the $(N+1)\times(N+1)$ Cauchy matrix with parameters $k_1,\ldots,k_{N+1}$ as defined in a similar way as in (2.1). Applying the lemma to compute the inverse of the matrix $\mathbf{1}+\boldsymbol{M}^{(N+1)}$, which can be decomposed as above by setting

$$\boldsymbol{A}=\mathbf{1}+\boldsymbol{M}^{(N)},\qquad \boldsymbol{b}=c_{N+1}(k_{N+1}\mathbf{1}+\boldsymbol{K})^{-1}\boldsymbol{r},$$
$$\boldsymbol{c}=\rho_{N+1}\boldsymbol{c}^T(k_{N+1}\mathbf{1}+\boldsymbol{K})^{-1},\qquad d=1+\frac{\rho_{N+1}c_{N+1}}{2k_{N+1}},$$

we have

$$(\mathbf{1}+\boldsymbol{M}^{(N+1)})^{-1}=\left(\begin{array}{c|c}(\mathbf{1}+\boldsymbol{M}^{(N)})^{-1}+s^{-1}c_{N+1}\rho_{N+1}\boldsymbol{u}(k_{N+1})\,{}^t\boldsymbol{u}(k_{N+1}) & -s^{-1}c_{N+1}\boldsymbol{u}(k_{N+1})\\ -s^{-1}\rho^t_{N+1}\boldsymbol{u}(k_{N+1}) & s^{-1}\end{array}\right),$$

where $\boldsymbol{u}(\cdot)$ and ${}^t\boldsymbol{u}(\cdot)$ are given in (2.34*a*) and (2.34*b*), and where s now takes the form

$$s=1+\frac{\rho_{N+1}c_{N+1}}{2k_{N+1}}(1-2k_{N+1}S^{(N)}(k_{N+1},k_{N+1}))=\frac{f^{(N+1)}}{f^{(N)}},\tag{4.3}$$

where $f^{(N+1)}=\det(\mathbf{1}+\boldsymbol{M}^{(N+1)})$ is the τ-function of the $(N+1)$-soliton solution. Using now definitions (2.35) and (2.32), we obtain the following recursion relations for the N to $N+1$ soliton objects:

$$S^{(N+1)}(a,b)=S^{(N)}(a,b)+\frac{s^{-1}c_{N+1}\rho_{N+1}}{(a+k_{N+1})(b+k_{N+1})}(1-(a+k_{N+1})S^{(N)}(a,k_{N+1}))$$
$$\times(1-(b+k_{N+1})S^{(N)}(k_{N+1},b)),\tag{4.4a}$$
$$V^{(N+1)}(a)=V^{(N)}(a)-\frac{s^{-1}c_{N+1}\rho_{N+1}}{a+k_{N+1}}(1-(a+k_{N+1})S^{(N)}(a,k_{N+1}))V^{(N)}(k_{N+1}),\tag{4.4b}$$

which hold for all a,b (not coinciding with $-k_i$, $i=1,\ldots,N$). Setting $b=k_{N+1}$ (assumed also not to coincide with any of the $-k_i$, $i=1,\ldots,N$) we obtain as a corollary of these relations the following identifications:

$$s=\frac{1-(a+k_{N+1})S^{(N)}(a,k_{N+1})}{1-(a+k_{N+1})S^{(N+1)}(a,k_N)}=\frac{V^{(N)}(k_{N+1})}{V^{(N+1)}(k_{N+1})},\tag{4.5}$$

which, in particular, implies that the ratio of $1-(a+k_{N+1})S(a,k_{N+1})$ between its N- and $N+1$-soliton values is independent of the parameter a.

Step # 2. In this step, we will apply the BT with the Bäcklund parameter $\mathfrak{l}=(l,L)\in\Gamma$ to the N-soliton solution defined in (3.9); i.e. we want to solve the system of discrete Riccati equations for a new variable v :

$$\mathcal{Q}_{\mathfrak{p},\mathfrak{l}}(u^{(N)},\widetilde{u}^{(N)},v,\widetilde{v})=0,\qquad \mathcal{Q}_{\mathfrak{q},\mathfrak{l}}(u^{(N)},\widehat{u}^{(N)},v,\widehat{v})=0,\tag{4.6}$$

cf [6]. To solve this system we can reduce the problem by identifying two particular solutions, which are obtained by covariant extension of the known solution $u^{(N)}$. By covariant extension we mean that all the plane-wave factors ρ_i $(i=1,\ldots,N)$, as well as the discrete exponential factors $\digamma_{n,m}(\pm a,\pm b)$, which enter in the N-soliton formula (e.g. through the Cauchy matrix $\boldsymbol{M}=\boldsymbol{M}^{(N)}$ and the vectors $\boldsymbol{r}$), have a dependence on an additional lattice variable h associated with the lattice parameter l, such that

$$\rho_i^0=\left(\frac{l+k_i}{l-k_i}\right)^h\rho_i^{00}\quad\Rightarrow\quad\overline{\rho}_i=\left(\frac{l+k_i}{l-k_i}\right)\rho_i,\tag{4.7a}$$

and

$$\digamma_{n,m,h}(\pm a,\pm b)=\digamma_{n,m}(\pm a,\pm b)\left(\frac{L}{(l-a)(l-b)}\right)^h$$
$$\Rightarrow\quad\overline{\digamma}(\pm a,\pm b)=\left(\frac{L}{(l\mp a)(l\mp b)}\right)\digamma(\pm a,\pm b).\tag{4.7b}$$

Here the elementary discrete shift in h is indicated by a $\bar{}$, which gives meaning to the expressions $\overline{u} = T_l u$, $\underline{u} = T_l^{-1} u$, i.e. for the thus covariantly extended solution $u_{n,m,h}^{(N)}$ we have

$$\overline{u}_{n,m,h}^{(N)} = u_{n,m,h+1}^{(N)}, \qquad \underline{u}_{n,m,h}^{(N)} = u_{n,m,h-1}^{(N)}.$$

As a consequence of the construction of theorem 1, the following equations are satisfied:

$$\mathcal{Q}_{\mathfrak{p},\mathfrak{l}}(u^{(N)}, \widetilde{u}^{(N)}, \overline{u}^{(N)}, \widetilde{\overline{u}}^{(N)}) = 0, \qquad \mathcal{Q}_{\mathfrak{q},\mathfrak{l}}(u^{(N)}, \widehat{u}^{(N)}, \overline{u}^{(N)}, \widehat{\overline{u}}^{(N)}) = 0, \tag{4.8a}$$

$$\mathcal{Q}_{\mathfrak{p},\mathfrak{l}}(u^{(N)}, \widetilde{u}^{(N)}, \underline{u}^{(N)}, \widetilde{\underline{u}}^{(N)}) = 0, \qquad \mathcal{Q}_{\mathfrak{q},\mathfrak{l}}(u^{(N)}, \widehat{u}^{(N)}, \underline{u}^{(N)}, \widehat{\underline{u}}^{(N)}) = 0, \tag{4.8b}$$

where (4.8*b*) holds because of the symmetry of the quadrilaterals. Having established the two solutions $\overline{u}^{(N)}$ and $\underline{u}^{(N)}$ of (4.6) we can now find the general solution of that system in the interpolating form:

$$v = \frac{\overline{u}^{(N)} + \eta \underline{u}^{(N)}}{1 + \eta}, \tag{4.9}$$

in which η is some function to be determined from the following coupled system of homogeneous linear equations:

$$\frac{\widetilde{\eta}}{\eta} = -\frac{\mathcal{Q}_{\mathfrak{p},\mathfrak{l}}(u^{(N)}, \widetilde{u}^{(N)}, \underline{u}^{(N)}, \widetilde{\overline{u}}^{(N)})}{\mathcal{Q}_{\mathfrak{p},\mathfrak{l}}(u^{(N)}, \widetilde{u}^{(N)}, \overline{u}^{(N)}, \widetilde{\underline{u}}^{(N)})}, \tag{4.10a}$$

$$\frac{\widehat{\eta}}{\eta} = -\frac{\mathcal{Q}_{\mathfrak{q},\mathfrak{l}}(u^{(N)}, \widehat{u}^{(N)}, \underline{u}^{(N)}, \widehat{\overline{u}}^{(N)})}{\mathcal{Q}_{\mathfrak{q},\mathfrak{l}}(u^{(N)}, \widehat{u}^{(N)}, \overline{u}^{(N)}, \widehat{\underline{u}}^{(N)})}. \tag{4.10b}$$

The compatibility of this system is equivalent to the compatibility of the BT (4.6) as a coupled system of discrete Riccati equations, which, in turn, is a consequence of the multidimensional consistency of the lattice equation. Using the explicit expressions for the quadrilaterals (3.26), equations (4.10) reduce to

$$\frac{\widetilde{\eta}}{\eta} = \frac{P\widetilde{\overline{u}}^{(N)} - (p^2 - l^2)u^{(N)} - L\underline{u}^{(N)}}{P\widetilde{\underline{u}}^{(N)} - (p^2 - l^2)u^{(N)} - L\overline{u}^{(N)}} = \frac{p + l + \underline{w} - \widetilde{w}}{p - l + \overline{w} - \widetilde{w}}, \tag{4.11a}$$

$$\frac{\widehat{\eta}}{\eta} = \frac{Q\widehat{\overline{u}}^{(N)} - (q^2 - l^2)u^{(N)} - L\underline{u}^{(N)}}{Q\widehat{\underline{u}}^{(N)} - (q^2 - l^2)u^{(N)} - L\overline{u}^{(N)}} = \frac{q + l + \underline{w} - \widehat{w}}{q - l + \overline{w} - \widehat{w}}, \tag{4.11b}$$

where in the last step we have made use of equation (3.12*a*) and the backward shifted version of (3.12*c*) in the relevant lattice directions. Finally, using equations (2.41*b*) and (2.41*c*) with $q \to l$ and $\widehat{w} \to \overline{w}$, etc, setting $a = l$, which lead to

$$p - l + \overline{w} - \widetilde{w} = (p - l)\frac{\overline{V}(l)}{\widetilde{V}(l)}, \qquad p + l + \underline{w} - \widetilde{w} = (p + l)\frac{\widetilde{V}(l)}{V(l)},$$

we get

$$\frac{\widetilde{\eta}}{\eta} = \left(\frac{p + l}{p - l}\right)\frac{\widetilde{V}(l)\widetilde{\overline{V}}(l)}{V(l)\overline{V}(l)}, \qquad \frac{\widehat{\eta}}{\eta} = \left(\frac{q + l}{q - l}\right)\frac{\widehat{V}(l)\widehat{\overline{V}}(l)}{V(l)\overline{V}(l)},$$

which can be simultaneously integrated, yielding the following expression for the function η:

$$\eta = \eta_{n,m} = \eta_0 \left(\frac{p + l}{p - l}\right)^n \left(\frac{q + l}{q - l}\right)^m V_{n,m}(l)\overline{V}_{n,m}(l). \tag{4.12}$$

Substituting expression (4.12) into (4.9) we have obtained the general solution of the system (4.6). It will be convenient to recast solution (4.9) in the following form:

$$v = \overline{u}^{(N)} - \frac{\eta}{1+\eta}(\overline{u}^{(N)} - \underline{u}^{(N)}) = \overline{u}^{(N)} - \left(\frac{\eta}{1+\eta}\right)\frac{1}{L}(2l + \underline{w} - \overline{w})U^{(N)}, \tag{4.13}$$

where we have inserted the expression for the difference $\overline{u}^{(N)} - \underline{u}^{(N)}$ which can be directly obtained from the backward-shifted relation of (3.12*c*) by taking $\mathfrak{p} = \mathfrak{q} = \mathfrak{l}$ and identifying the shifts $\widetilde{w}$ and $\widehat{w}$ (and similarly the shifts on u and U) with the shift $\overline{w}$ associated with the parameter $\mathfrak{l}$.

Step # 3. We now set the Bäcklund parameter $\mathfrak{l} = \mathfrak{k}_{N+1} = (k_{N+1}, K_{N+1})$, which implies that the $^{-}$-shift from now on is the lattice shift associated with the lattice parameter $\mathfrak{k}_{N+1}$. This enables us to reexpress the factors in (4.4*a*) in the following way:

$$1 - (k_{N+1} + a)S^{(N)}(a, k_{N+1}) = V^{(N)}(k_{N+1})\underline{V}^{(N)}(a), \tag{4.14}$$

which is an identity that holds for all a, and which follows from

$$1 + (l - b)\underline{S}^{(N)}(a, b) - (l + a)S^{(N)}(a, b) = \underline{V}^{(N)}(a)V^{(N)}(b), \tag{4.15}$$

by setting $l = b = k_{N+1}$, which, in turn, is a covariantly extended version of (2.37) in the lattice direction associated with the parameter $\mathfrak{l}$. Using the identity (4.14) in (4.4*a*) and inserting the result into expression (3.9) for $u^{(N+1)}$ we find

$$\begin{aligned}
u^{(N+1)} &= AF(a,b)[1 - (a+b)S^{(N+1)}(a,b)] + \cdots \\
&= AF(a,b)\Bigg[(1 - (a+b)S^{(N)}(a,b)) \\
&\quad - \frac{(a+b)\rho_{N+1}c_{N+1}s^{-1}}{(a+k_{N+1})(b+k_{N+1})}(V^{(N)}(k_{N+1}))^2\underline{V}^{(N)}(a)\underline{V}^{(N)}(b)\Bigg] + \cdots \\
&= u^{(N)} - \frac{1}{K_{N+1}}(V^{(N)}(k_{N+1}))^2\rho_{N+1}c_{N+1}s^{-1}[A(a+b)\underline{F}(a,b)\underline{V}^{(N)}(a)\underline{V}^{(N)}(b) + \cdots],
\end{aligned}$$

where again the dots stand for the remaining terms with coefficients B, C, D and with (a, b) replaced by $(a, -b), (-a, b), (-a, -b)$, respectively, leading to the following formula for the $N+1$-soliton solution:

$$u^{(N+1)} = u^{(N)} - \frac{1}{K_{N+1}}(V^{(N)}(k_{N+1}))^2\rho_{N+1}c_{N+1}s^{-1}\underline{U}^{(N)}. \tag{4.16}$$

On the other hand, setting $\mathfrak{l} = \mathfrak{k}_{N+1}$, and choosing the constant $\eta_0 = c_{N+1}/(2k_{N+1})$ in (4.12), we obtain the identification

$$1 + \eta = \overline{s}, \tag{4.17}$$

with s given in (4.3), using also (4.14) in the case of $a = k_{N+1}$. Inserting (4.17) into the solution v of the BT, (4.13), with $\mathfrak{l} = \mathfrak{k}_{N+1}$, and using the relation

$$2k_{N+1} + \overline{w} - \underline{w} = 2k_{N+1}\frac{\overline{V}^{(N)}(k_{N+1})}{V^{(N)}(k_{N+1})},$$

which follows from the backward shift of (2.41*c*) setting $p = q = a = k_{N+1}$, we can make the identification $v = \overline{u}^{(N+1)}$ between (4.13) and the forward shift of (4.16). This establishes relation (4.2), and hence completes the proof of theorem 2. □

Theorem 2 establishes the precise connection between the structure of the N-soliton solution as given by the Cauchy matrix approach, and the way to generate a soliton hierarchy through BTs. What we conclude is that these two approaches coincide up to a subtle identification of the relevant constants in the solution. Since, as was remarked in the corollary of section 3, the soliton solutions of Q3 really live in an extended four-dimensional lattice, the precise identification of those constants is of interest, since they contain possibly the additional lattice directions. In fact, in [6] we established the first soliton-type solutions for Q4 through the Bäcklund approach, and it is of interest to see how that approach connects to a (yet unknown) representation of multi-soliton solutions in terms of a scheme similar to that set up in this paper for Q3.

4.2. Connection to the Bäcklund transformation fixed point

In [6], we introduced a method for the construction of an elementary solution (of a multidimensionally consistent lattice equation) which is suitable as a seed for the subsequent construction of soliton solutions by the iterative application of the BT. This method is based on an idea which can actually be traced back to Weiss in [31], cf [8] for a further discussion of this point. We will now describe the connection between the solution which arises when that method is applied to construct a seed solution for Q3 (3.7) and the 0-soliton solution found by the substitution of $N = 0$ into the principal object studied here (3.9). Note that when we substitute $N = 0$ in (3.9) the functions $S(\pm a, \pm b) = S_{n,m}(\pm a, \pm b)$ appearing in the solution all vanish identically.

Following [6], we consider a solution of (3.7) $u_\theta = (u_\theta)_{n,m}$ which is related to itself by the BT of (3.7) with the Bäcklund parameter $\mathfrak{t} \in \Gamma$ (recall Γ was defined in (3.6)). That is, u_θ satisfies the system:

$$\mathcal{Q}_{\mathfrak{p},\mathfrak{t}}(u_\theta, \widetilde{u}_\theta, u_\theta, \widetilde{u}_\theta) = 0, \qquad \mathcal{Q}_{\mathfrak{q},\mathfrak{t}}(u_\theta, \widehat{u}_\theta, u_\theta, \widehat{u}_\theta) = 0. \tag{4.18}$$

These are coupled biquadratic equations for u_θ which are parameter deformations of the biquadratic introduced earlier (3.27). In this case, we need to introduce a limit on the curve, $\Gamma \ni \mathfrak{t} = (t, T) \longrightarrow \infty^+$ in which $t \longrightarrow \infty$ and $T \to t^2 - a^2/2 - b^2/2 + O(t^{-2})$, then the identity,

$$\lim_{\mathfrak{t}\longrightarrow\infty^+} \mathcal{Q}_{\mathfrak{p},\mathfrak{t}}(u, \widetilde{u}, u, \widetilde{u}) = -\mathcal{H}_{\mathfrak{p}}(u, \widetilde{u}), \tag{4.19}$$

reveals the sense in which the one is a deformation of the other. Recall [6] that with this special choice of $\mathfrak{t}$ the system (4.18) defines a *non-germinating* seed solution (or *singular* solution in [4]).

To solve simultaneously the biquadratic equations (4.18) (for general $\mathfrak{t}$) we introduce parameters p_θ and q_θ defined in terms of $\mathfrak{p}$ and $\mathfrak{q}$ by the quadratic equations

$$p_\theta + 1/p_\theta = 2\frac{T - t^2 + p^2}{P}, \qquad q_\theta + 1/q_\theta = 2\frac{T - t^2 + q^2}{Q}. \tag{4.20}$$

The (canonical) solution of (4.18) may then be written:

$$u_\theta = (u_\theta)_{n,m} = A_\theta p_\theta^n q_\theta^m + B_\theta p_\theta^{-n} q_\theta^{-m}, \tag{4.21}$$

where the coefficients A_θ and B_θ are constants subject to the single constraint

$$A_\theta B_\theta = \frac{\delta^2}{16T(2t^2 - 2T - a^2 - b^2)}. \tag{4.22}$$

We are now in a position to describe the connection between the solution found as a fixed point of the BT (4.21) and the solution $u^{(0)}$, i.e., the solution obtained by setting $N = 0$ in (3.9). Whereas that solution contains four terms with coefficients A, B, C and D, solution (4.21) contains only two terms, so it is clear that the solutions do not coincide. Actually, what

$$
\begin{tikzcd}
\boxed{\text{Q3}} \arrow[r] \arrow[d] & \boxed{\text{Q2}} \arrow[r] \arrow[d] & \boxed{\text{Q1}} \arrow[d] \\
\boxed{\text{H3}} \arrow[r] & \boxed{\text{H2}} \arrow[r] & \boxed{\text{H1}}
\end{tikzcd}
$$

Figure 3. Coalescence scheme employed to construct N-soliton solutions for the degenerate subcases of equation Q3.

we find is that making the particular choice $\mathfrak{t} = (0, ab)$ solution (4.21) becomes $u^{(0)}$ with $A = A_\theta$, $D = B_\theta$ and $B = C = 0$, whilst making the choice $\mathfrak{t} = (0, -ab)$ (4.21) becomes $u^{(0)}$ with $A = D = 0$, $B = A_\theta$ and $C = B_\theta$. In both cases the constraint (4.22) becomes (3.10).

5. *N*-soliton solutions for the degenerate subcases of Q3

The coalescence scheme illustrated in figure 3 will be used in this section to construct N-soliton solutions for the equations Q2, Q1, H3, H2 and H1 by degeneration from the N-soliton solution we have given for Q3. We begin by detailing how the equations themselves are found by degeneration, then we show how to find the new solutions.

5.1. The degenerations from Q3

We degenerate from Q3 as parametrized in (3.7) which differs from the parametrization given originally by ABS [3] (which we reproduce in (1.1*c*)). The degenerate equations consequently appear in a parametrization different from the lists (1.1) and (1.2). Importantly, throughout the scheme depicted in figure 3 there is no limit taken on the parameters p and q present in (3.7), so each equation emerges in terms of these *common* lattice parameters (which later will be seen to occur naturally in all of the N-soliton solutions). These parameters are simply related to the parameters $\mathring{p}$ and $\mathring{q}$ of the equations listed in (1.1) and (1.2) by the following associations:

$$\text{Q3}: \qquad \mathring{p} = \frac{P}{p^2 - a^2} = \frac{p^2 - b^2}{P}, \qquad \mathring{q} = \frac{Q}{q^2 - a^2} = \frac{q^2 - b^2}{Q}, \tag{5.1a}$$

$$\text{Q2, Q1}: \qquad \mathring{p} = \frac{a^2}{p^2 - a^2}, \qquad \mathring{q} = \frac{a^2}{q^2 - a^2}, \tag{5.1b}$$

$$\text{H3}: \qquad \mathring{p} = \frac{P}{a^2 - p^2} = \frac{1}{P}, \qquad \mathring{q} = \frac{Q}{a^2 - q^2} = \frac{1}{Q}, \tag{5.1c}$$

$$\text{H2, H1}: \qquad \mathring{p} = -p^2, \qquad \mathring{q} = -q^2. \tag{5.1d}$$

When written in terms of these common parameters p and q, the equations Q3 and H3 involve also upper-case parameters P and Q which are defined in terms of p and q by an algebraic relation. For Q3, this relation was introduced previously (3.4) and is given again in (5.1*a*); for H3 there is a different algebraic relation which is given in (5.1*c*).

It is by limits on the parameters a and b and on the dependent variable u that the degenerations appearing in figure 3 are achieved. Specifically, the following list of substitutions results in the indicated degenerations in the limit $\epsilon \longrightarrow 0$:

$$\mathrm{Q3} \longrightarrow \mathrm{Q2} : b = a(1-2\epsilon), \qquad u \to \frac{\delta}{4a^2}\left(\frac{1}{\epsilon} + 1 + (1+2u)\epsilon\right), \tag{5.2a}$$

$$\mathrm{Q2} \longrightarrow \mathrm{Q1} : \qquad u \to \frac{\delta^2}{4\epsilon^2} + \frac{1}{\epsilon}u, \tag{5.2b}$$

$$\mathrm{Q3} \longrightarrow \mathrm{H3} : b = \frac{1}{\epsilon^2}, \qquad u \to \frac{\sqrt{\delta}}{2}\epsilon^3 u, \tag{5.2c}$$

$$\mathrm{Q2} \longrightarrow \mathrm{H2} : a = \frac{1}{\epsilon}, \qquad u \to \frac{1}{4} + \epsilon^2 u, \tag{5.2d}$$

$$\mathrm{Q1} \longrightarrow \mathrm{H1} : a = \frac{1}{\epsilon}, \qquad u \to \epsilon\delta u, \tag{5.2e}$$

$$\mathrm{H3} \longrightarrow \mathrm{H2} : a = \frac{1}{\epsilon^2}, \qquad u \to \sqrt{-\delta}\epsilon\left(1 + \frac{\epsilon^4}{2}u\right), \tag{5.2f}$$

$$\mathrm{H2} \longrightarrow \mathrm{H1} : \qquad u \to \frac{1}{\epsilon^2} + \frac{2}{\epsilon}u. \tag{5.2g}$$

Note that for these degenerations we assume the parameter δ appearing in the equations Q3, Q1 and H3 is nonzero.

5.2. N-soliton solutions of Q2 and Q1

We degenerate from the previously established N-soliton solution for Q3 (3.9). To begin we will consider, in detail, the degeneration from this solution to the N-soliton solution of Q2. We are led by the requirement that the parameter b which appears in this solution should, according to (5.2a), be replaced with $b = a(1-2\epsilon)$. Making this substitution and expanding the result in powers of ϵ results in a lengthy expansion of (3.9) which we break down into expansions of the simpler component parts. First we consider the function $F(a, b)$ defined in (3.3) which has the following expansions depending on its choice of argument:

$$\begin{aligned}
F(a,b) &\longrightarrow \rho(a)(1 + \epsilon\xi + \epsilon^2(\xi^2/2 + \chi)) + O(\epsilon^3),\\
F(a,-b) &\longrightarrow 1 - \epsilon\xi + \epsilon^2(\xi^2/2 - \chi) + O(\epsilon^3),\\
F(-a,b) &\longrightarrow 1 + \epsilon\xi + \epsilon^2(\xi^2/2 + \chi) + O(\epsilon^3),\\
F(-a,-b) &\longrightarrow \rho(-a)(1 - \epsilon\xi + \epsilon^2(\xi^2/2 - \chi)) + O(\epsilon^3),
\end{aligned} \tag{5.3}$$

in which we have introduced the new functions

$$\begin{aligned}
\xi = \xi_{n,m} &= 2a\left(\frac{p}{a^2-p^2}n + \frac{q}{a^2-q^2}m\right),\\
\chi = \chi_{n,m} &= 4a^3\left(\frac{p}{(a^2-p^2)^2}n + \frac{q}{(a^2-q^2)^2}m\right),
\end{aligned} \tag{5.4}$$

and the function $\rho(a) = \rho_{n,m}(a)$ coincides with that defined previously (3.17). In a similar fashion, we find the expansion of terms involving $S(a, b)$ defined in (2.34) to be the following:

$$\begin{aligned}
1-(a+b)S(a,b) &\longrightarrow 1 - 2aS(a,a) + O(\epsilon),\\
1-(a-b)S(a,-b) &\longrightarrow 1 - \epsilon 2aS(a,-a) + \epsilon^2 4a^2 Z(a,-a) + O(\epsilon^3),\\
1+(a-b)S(-a,b) &\longrightarrow 1 + \epsilon 2aS(-a,a) + \epsilon^2 4a^2 Z(-a,a) + O(\epsilon^3),\\
1+(a+b)S(-a,-b) &\longrightarrow 1 + 2aS(-a,-a) + O(\epsilon),
\end{aligned} \tag{5.5}$$

in which we have introduced the new quantity

$$Z(a,b) = {}^t\boldsymbol{c}(b\mathbf{1}+\boldsymbol{K})^{-2}\boldsymbol{u}(a) = {}^t\boldsymbol{c}(b\mathbf{1}+\boldsymbol{K})^{-2}(1+\boldsymbol{M})^{-1}(a\mathbf{1}+\boldsymbol{K})^{-1}\boldsymbol{r}. \tag{5.6}$$

Now, although the expansions of these components of (3.9) are fixed, by choosing the dependence of the constants A, B, C and D in (3.9) on the small parameter ϵ we can exert some control over its overall expansion. In order that it be of the required form, namely $u^{(N)} \longrightarrow \frac{\delta}{4a^2}\left(\frac{1}{\epsilon}+1+(1+2u^{(N)})\epsilon\right)$ as listed in (5.2*a*), we make the following choices for these constants:

$$\begin{aligned} A &\to \frac{\delta}{4a^2}A\epsilon,\\ B &\to \frac{\delta}{8a^2}\left(\frac{1}{\epsilon}+1-\xi_0+\left((3+\xi_0^2)/2+2AD\right)\epsilon\right),\\ C &\to \frac{\delta}{8a^2}\left(\frac{1}{\epsilon}+1+\xi_0+\left((3+\xi_0^2)/2+2AD\right)\epsilon\right),\\ D &\to \frac{\delta}{4a^2}D\epsilon. \end{aligned} \tag{5.7}$$

Here the four constants A, B, C and D (constrained by (3.10)) which are present in the Q3 N-soliton solution (3.9) have been replaced with three modified constants A, D and ξ_0 (with no constraint), in the degeneration to the Q2 N-soliton solution. The number of these constants minus the number of constraints is therefore preserved in the degeneration, which is strong evidence that the solution found by degeneration is the most general one obtainable by this method. Finally then, the solution of Q2 which arises in the limit $\epsilon \longrightarrow 0$ as a consequence of the substitutions (5.3), (5.5) and (5.7) into (3.9) is found to be

$$\begin{aligned} u^{(N)} = {}& \tfrac{1}{4}((\xi+\xi_0)^2+1) + a(\xi+\xi_0)S(-a,a) + a^2\left(Z(a,-a)+Z(-a,a)\right)\\ &+ AD + \tfrac{1}{2}A\rho(a)(1-2aS(a,a)) + \tfrac{1}{2}D\rho(-a)(1+2aS(-a,-a)). \end{aligned} \tag{5.8}$$

Here ξ_0, A and D are the aforementioned constants which may be chosen arbitrarily, $\xi = \xi_{n,m}$ and $\rho(a) = \rho_{n,m}(a)$ are defined in (5.4) and (3.17), and $S(a,b) = S_{n,m}(a,b)$ and $Z(a,b) = Z_{n,m}(a,b)$ are defined in (2.34) and (5.6), respectively.

We now consider the degeneration Q2 $\longrightarrow$ Q1, which is somewhat simpler than the Q3 $\longrightarrow$ Q2 degeneration considered above. To achieve the required limit (5.2*b*) of the Q2 N-soliton solution (5.8), namely that $u^{(N)} \longrightarrow \frac{\delta^2}{4\epsilon^2}+\frac{1}{\epsilon}u^{(N)}$, we need only substitute the constants appearing in the solution as follows:

$$A \to \frac{2A}{\epsilon}, \qquad D \to \frac{2D}{\epsilon}, \qquad \xi_0 \to \xi_0 + \frac{2B}{\epsilon}. \tag{5.9}$$

The desired limit results from the substitution of (5.9) into (5.8), provided the modified constants A, B, D and ξ_0 are chosen to satisfy the single constraint

$$AD + \frac{1}{4}B^2 = \frac{\delta^2}{16}. \tag{5.10}$$

Note that again the number of constants minus constraints is preserved in the Q2 $\longrightarrow$ Q1 degeneration. The N-soliton solution of Q1 which emerges is

$$u^{(N)} = A\rho(a)(1-2aS(a,a)) + B(\xi+\xi_0+2aS(-a,a)) + D\rho(-a)(1+2aS(-a,-a)). \tag{5.11}$$

As a small side development here we now consider the further limit $a \longrightarrow 0$ in the Q1 N-soliton solution (5.11). There is some subtlety in this limit and the solution of Q1 which emerges inspires some useful additional observations. Performing the $a \longrightarrow 0$ limit naively would also change the equation (in fact send it to $\mathrm{Q1}_{\delta=0}$) because a appears in its parametrization (5.1*b*). However, the substitution,

$$a = \epsilon, \qquad u \to 1 + \epsilon^2 u, \tag{5.12}$$

preserves the full equation as $\epsilon \longrightarrow 0$. This leads to a reparametrization of Q1 in which we identify the parameters $\mathring{p}$ and $\mathring{q}$ present in (1.1*a*) simply as

$$\mathring{p} = \frac{1}{p^2}, \qquad \mathring{q} = \frac{1}{q^2}, \tag{5.13}$$

which come to replace the associations in (5.1*b*). Now the same substitution, $a = \epsilon$, yields the following small-ϵ expansion for the component parts of solution (5.11):

$$\begin{aligned}
&\rho(a) \longrightarrow 1 + \epsilon\nu + \epsilon^2\frac{\nu^2}{2} + O(\epsilon^3),\\
&\rho(-a) \longrightarrow 1 - \epsilon\nu + \epsilon^2\frac{\nu^2}{2} + O(\epsilon^3),\\
&\xi \longrightarrow -\epsilon\nu + O(\epsilon^3),\\
&S(a,a) \longrightarrow S^{(-1,-1)} - 2\epsilon S^{(-1,-2)} + O(\epsilon^2),\\
&S(-a,a) \longrightarrow S^{(-1,-1)} + O(\epsilon^2),\\
&S(-a,-a) \longrightarrow S^{(-1,-1)} + 2\epsilon S^{(-1,-2)} + O(\epsilon^2),
\end{aligned} \tag{5.14}$$

where we have introduced the new function ν,

$$\nu = \nu_{n,m} = \frac{2}{p}n + \frac{2}{q}m. \tag{5.15}$$

If we also make the choice

$$\begin{aligned}
&A \to \frac{1}{2} + \frac{\epsilon}{2}(A + \nu_0),\\
&B \to 1 - \frac{\delta}{2} + \frac{\epsilon}{2}((1-\delta)\nu_0 - 2A\nu_1/\nu_0),\\
&D \to \frac{\delta - 1}{2} + \frac{\epsilon}{2}A(2\nu_1/\nu_0 - 1),\\
&\xi_0 \to 1 - \epsilon\nu_0 + \frac{\epsilon^2}{2}\nu_0^2,
\end{aligned} \tag{5.16}$$

for the constants appearing in (5.11), then the substitution of (5.14) yields the desired limit of the solution $u^{(N)} \longrightarrow 1 + \epsilon^2 u^{(N)}$. The resulting solution of Q1 is as follows:

$$u^{(N)} = \delta\left(\tfrac{1}{4}(\nu+\nu_0)^2 - (\nu+\nu_0)S^{(-1,-1)} + 2S^{(-1,-2)}\right) + A(\nu + \nu_1 - 2S^{(-1,-1)}), \tag{5.17}$$

where the constants ν_0, ν_1 and A are arbitrary, $\nu = \nu_{n,m}$ is defined in (5.15) and $S^{(i,j)} = S^{(i,j)}_{n,m}$ is defined in (2.8*c*).

The main point we wish to make about the Q1 solution (5.17) is that it generalizes the solution previously given (2.30) for the lattice Schwarzian KdV equation, i.e., the equation $\mathrm{Q1}_{\delta=0}$. Specifically this can be seen as an extension of that solution to the case $\delta \neq 0$ in that it reduces to (2.30) if we take $\delta = 0$ and $A = -1/2$.

5.3. N-soliton solution of H3

To find the N-soliton solution for H3 we degenerate from the Q3 N-soliton solution (3.9) led now by the requirement that, according to (5.2*c*), we choose $b = \frac{1}{\epsilon^2}$. As before we give small-ϵ expansions for the component parts of the Q3 N-soliton solution (3.9) which result from making this substitution for b. We find that

$$\begin{aligned}
&F(a,b) \longrightarrow \vartheta + O(\epsilon^2),\\
&F(a,-b) \longrightarrow (-1)^{n+m}\vartheta + O(\epsilon^2),\\
&F(-a,b) \longrightarrow (-1)^{n+m}\vartheta^{-1} + O(\epsilon^2),\\
&F(-a,-b) \longrightarrow \vartheta^{-1} + O(\epsilon^2),\\
&1-(a+b)S(a,b) \longrightarrow V(a) + O(\epsilon^2),\\
&1-(a-b)S(a,-b) \longrightarrow V(a) + O(\epsilon^2),\\
&1+(a-b)S(-a,b) \longrightarrow V(-a) + O(\epsilon^2),\\
&1+(a+b)S(-a,-b) \longrightarrow V(-a) + O(\epsilon^2),
\end{aligned} \tag{5.18}$$

where we have introduced the new function

$$\vartheta = \vartheta_{n,m} = \left(\frac{P}{a-p}\right)^n \left(\frac{Q}{a-q}\right)^m, \tag{5.19}$$

which involves parameters P and Q which are related to p and q by (5.1c), and where $V(a) = V_{n,m}(a)$ is defined in (2.32).

Substituting expressions (5.18) into (3.9) whilst choosing the constants in that solution to be

$$\begin{aligned}
&A \to \epsilon^3 \tfrac{\sqrt{\delta}}{2} A, \quad B \to \epsilon^3 \frac{\sqrt{\delta}}{2} B,\\
&C \to \epsilon^3 \tfrac{\sqrt{\delta}}{2} C, \quad D \to \epsilon^3 \frac{\sqrt{\delta}}{2} D,
\end{aligned} \tag{5.20}$$

we find $u^{(N)} \longrightarrow \epsilon^3 \frac{\sqrt{\delta}}{2} u^{(N)}$ as required for the Q3$\longrightarrow$ H3 limit given in (5.2c). Thus we find the N-soliton solution of H3 to be

$$u^{(N)} = (A + (-1)^{n+m}B)\vartheta V(a) + ((-1)^{n+m}C + D)\vartheta^{-1}V(-a). \tag{5.21}$$

Here the constants A, B, C and D are subject to the single constraint

$$AD - BC = \frac{-\delta}{4a},$$

which follows by the substitution of $b = \frac{1}{\epsilon^2}$ and (5.20) into (3.10). The functions $\vartheta = \vartheta_{n,m}$ and $V(a) = V_{n,m}(a)$ are defined in (5.19) and (2.32).

5.4. N-soliton solutions of H2 and H1

To find the H2 N-soliton solution we choose to degenerate from the Q2 N-soliton solution (5.8) led by the requirement that we substitute $a = \frac{1}{\epsilon}$ (cf (5.2d)). (Observe from figure 3 that we could choose to degenerate from the H3 N-soliton solution (5.21) to the H2 N-soliton solution; the two paths actually lead to the same result.) Making the substitution $a = \frac{1}{\epsilon}$ into the component parts of (5.8) yields the following small-ϵ expansions:

$$\begin{aligned}
&\xi \longrightarrow \epsilon\zeta + O(\epsilon^3),\\
&\rho(a) \longrightarrow (-1)^{n+m}(1 + \epsilon\zeta + \epsilon^2\zeta^2/2 + O(\epsilon^3)),\\
&\rho(-a) \longrightarrow (-1)^{n+m}(1 - \epsilon\zeta + \epsilon^2\zeta^2/2 + O(\epsilon^3)),
\end{aligned} \tag{5.22}$$

in which we have introduced a new function ζ,

$$\zeta = \zeta_{n,m} = 2np + 2mq, \tag{5.23}$$

and

$$\begin{aligned}
&aS(-a,a) \longrightarrow -\epsilon S^{(0,0)} + O(\epsilon^2),\\
&aS(a,a) \longrightarrow \epsilon S^{(0,0)} - 2\epsilon^2 S^{(0,1)} + O(\epsilon^3),\\
&aS(-a,-a) \longrightarrow \epsilon S^{(0,0)} + 2\epsilon^2 S^{(0,1)} + O(\epsilon^3),\\
&a^2(Z(-a,a)+Z(a,-a)) \longrightarrow 2\epsilon^2 S^{(0,1)} + O(\epsilon^3).
\end{aligned} \tag{5.24}$$

Substituting (5.22) and (5.24) into (5.8) combined with the following choice for the constants,

$$\begin{aligned}
&A \to A(\epsilon + \zeta_1\epsilon^2/2),\\
&D \to A(-\epsilon + \zeta_1\epsilon^2/2),\\
&\xi_0 \to \epsilon\zeta_0,
\end{aligned} \tag{5.25}$$

results in an expansion for (5.8) of the required form $u^{(N)} \longrightarrow \frac{1}{4} + \epsilon^2 u^{(N)}$ (cf (5.2*d*)) with the new H2 N-soliton solution which results being

$$u^{(N)} = \frac{1}{4}(\zeta+\zeta_0)^2 - (\zeta+\zeta_0)S^{(0,0)} + 2S^{(0,1)} - A^2 + (-1)^{n+m}A(\zeta+\zeta_1-2S^{(0,0)}), \tag{5.26}$$

where the constants A, ζ_0 and ζ_1 are arbitrary (and unrelated), $\zeta = \zeta_{n,m}$ is defined in (5.23) and $S^{(i,j)} = S^{(i,j)}_{n,m}$ is defined in (2.8*c*).

We remark that solution (5.26) in the case $A = 0$ can be transformed to the solution given previously for the equation Q1 (5.17) with $A = 0$ and $\delta = 1$ by the simple (self-inverse) transformation:

$$p, q, k_1 \ldots k_N \to 1/p, 1/q, 1/k_1 \ldots 1/k_N, \qquad \rho_i \to (-1)^{n+m}\rho_i. \tag{5.27}$$

The connection between these solutions reflects a kind of duality between the equations $\mathrm{Q1}_{\delta=1}$ and H2 which was found previously in [9].

To find the N-soliton solution for the equation H1 we choose to degenerate from the N-soliton solution of Q1 (5.11). According to (5.2*e*) we should make the substitution $a = \frac{1}{\epsilon}$ into (5.11), conveniently we have already expanded the component parts of this solution in powers of ϵ because they appeared in our consideration of the Q2 $\longrightarrow$ H2 degeneration detailed above in (5.22) and (5.24). The required limit of the solution, which according to (5.2*e*) is $u^{(N)} \longrightarrow \epsilon\delta u^{(N)}$, is achieved by choosing the constants appearing in (5.11) as follows:

$$\begin{aligned}
&A \to \frac{\delta}{2}A(1+\zeta_1\epsilon),\\
&D \to \frac{\delta}{2}A(-1+\zeta_1\epsilon),\\
&B \to \delta B,\\
&\xi_0 \to \epsilon\zeta_0.
\end{aligned} \tag{5.28}$$

The resulting N-soliton solution of H1 reads

$$u^{(N)} = B(\zeta+\zeta_0-2S^{(0,0)}) + (-1)^{n+m}A(\zeta+\zeta_1-2S^{(0,0)}), \tag{5.29}$$

where ζ_0, ζ_1, A and B, subject to the single constraint

$$A^2 - B^2 = \frac{-1}{4},$$

are otherwise arbitrary constants, $\zeta = \zeta_{n,m}$ and $S^{(i,j)} = S^{(i,j)}{}_{n,m}$ are defined in (5.23) and (2.8*c*), respectively.

5.5. Equations A2 and A1

In the above, we have constructed solutions by degeneration following the coalescence diagram of figure 3. This diagram does not include the equations A2 and A1, and we have thus far not given explicitly their *N*-soliton solutions. However, these equations are related to $Q3_{\delta=0}$ and Q1, respectively, by straightforward gauge transformation, so the solutions we have given for those equations may be transformed to solutions for the equations A2 and A1.

6. Concluding remarks

In this paper, we have reviewed the construction of *N*-soliton solution for integrable quadrilateral lattice equations of KdV type dating back to the early 1980s, cf [23, 28], as well as constructed *N*-soliton solutions for the majority of equations in the ABS list. We have concentrated particularly on the case of Q3, which by degeneration yields all the other ABS equations (except Q4) by limits on the parameters. The *N*-soliton solution of Q3 is particularly interesting as it is most conveniently described in a four-dimensional lattice, associated with four lattice parameters, two of which are the lattice parameters of the equation supplemented by two further parameters acting as branch points of an elliptic curve. The emergence of this elliptic curve is rather mysterious at the level of Q3, which, unlike Q4, does not really warrant an elliptic parametrization. Nonetheless, this curve naturally arises through what we consider to be the *universal* parametrization of all ABS equations (except perhaps Q4, which remains to be investigated), and which allows us to treat all equations in the list on the same footing (unlike the original parametrization from [3] where the parameters of the different equations in the list do not seem to be directly linked to each other). Thus, the *N*-soliton solution of Q3 can be written as a linear combination of four terms each of which contains as an essential ingredient the *N*-soliton solution of the so-called NQC equation of [23] with different values of the branch point parameters which enter in that equation. The *N*-soliton solutions of the other equations in the list, namely Q2, Q1, H3, H2, H1, follow by degeneration, and were derived in explicit form from the solutions of Q3. The present paper concentrated on an approach using a Cauchy matrix representation of the basic objects, and which essentially was developed as a *direct linearization approach* in the paper of the early 1980s. As a remarkable upshot, a novel Miura transform between Q3 (almost at the top of the ABS list) and H1 (at the bottom of the list) played a crucial role in the mechanism behind the solutions. In the subsequent paper [16], an alternative representation of the soliton solutions, in terms of Casorati determinants, is given, based on the bilinear forms of the ABS equations.

We have not touched in this paper on continuum limits of the equations and the hierarchies of continuous equations associated with the lattice systems. This can obviously be done without any problem. The most direct way of introducing continuum analogues is by simply replacing the discrete plane-wave factors ρ_i, which as functions of the discrete variables n, m were given in (2.2) by exponentials, making the replacements

$$\rho_i = \left(\frac{p+k_i}{p-k_i}\right)^n \left(\frac{q+k_i}{q-k_i}\right)^m \to \mathrm{e}^{2k_i x + 2k_i^3 t} \qquad \text{and} \qquad \digamma(a,b) \to \mathrm{e}^{(a+b)x+(a^3+b^3)t},$$

or by simply including the exponentials in the ρ_i together with the discrete exponential factors (invoking once again multidimensional consistency). We leave the derivation of the corresponding PDEs as an exercise for the reader.

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